Simplifying algebra in Feynman graphs. III. Massive vectors

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(Received 4 January 2001; published 29 May 2001)

A T-dualized self-dual inspired formulation of massive vector fields coupled to arbitrary matter is generated; subsequently its perturbative series modeling a spontaneously broken gauge theory is analyzed. The new Feynman rules and external line factors are chirally minimized in the sense that only one type of spin index occurs in the rules. Several processes are examined in detail and the cross sections formulated in this approach. A double line formulation of the Lorentz algebra for Feynman diagrams is produced in this formalism, similar to color ordering, which follows from a spin ordering of the Feynman rules. The new double line formalism leads to further minimization of gauge invariant scattering in perturbation theory. The dualized electroweak model is also generated.

DOI: 10.1103/PhysRevD.63.125027 PACS number(s): 11.15.Bt, 11.15.Pg, 11.55.−m

I. INTRODUCTION

Much progress has been made in the development of new tools useful for performing perturbative gauge theory calculations at high orders. These techniques include string inspired ones [1,2], color ordering [3,4], spinor helicity [5–7], constraints based on analyticity [8,9], unitarity methods [4], and those based on self-dual field theories [10–13]. These methods have enabled the calculation of several higher-point amplitudes necessary for next-to-leading order phenomenology as well as several sequences of helicity amplitudes with an arbitrary number of external legs and in multiple dimensions.[1]

In [22] a second-order formulation of fermionic couplings is derived by giving a chiral reduction of the minimally coupled theory. Integrating out the $\bar{\psi}^{\dot\alpha}$ and $\bar{\chi}^{\dot\alpha}$ components in the minimally coupled action $S=\mathrm{Tr}\int d^4x\mathcal{L}$ (in the conventions of Ref. [23]),

$$\mathcal{L}=-\bar{\psi}^{\dot\alpha} i\nabla_{\alpha\dot\alpha}\psi^{\alpha}-\bar{\chi}^{\dot\alpha} i\nabla_{\alpha\dot\alpha}\chi^{\alpha}+m(\psi^{\alpha}\chi_{\alpha}+\bar{\psi}^{\dot\alpha}\bar{\chi}_{\dot\alpha}) \qquad (1.1)$$

leads to a Lagrangian

$$\tilde{\mathcal{L}}=-\psi^{\alpha}(\Box-m^2)\chi_{\alpha}+\psi^{\alpha}F_{\alpha}{}^{\beta}\chi_{\beta}, \qquad (1.2)$$

and a gauge covariantized theory where (1) only the self-dual component of the gauge field strength couples to the fermions, and (2) in the remaining couplings the fermions are effectively bosonized. The new Lagrangian leads to a reduction in the amount of algebra one normally encounters in the computation of amplitudes because, for one reason, of the elimination of the gamma matrices directly at the level of the Lagrangian. Furthermore, because of the coupling only to the self-dual field strength [10], in maximally helicity violating (MHV) amplitudes only the first terms in Eq. (1.2) contribute and a perturbation around MHV structure is obtained through insertions of the latter terms. This is seen from the Lagrangian in Eq. (1.2) from the spin independence when truncating to $F_{\alpha\beta}=0$ (i.e., for amplitudes between states of the same helicity) [11]. Because of the spin independence of the couplings in this case, known supersymmetry identities [24,14] that these amplitudes obey become trivial. These properties are also a consequence of spectral flow in the $\mathcal{N}=2$ string theories [25].

In [12] we expanded and generalized these tools to the case of all partons, with spin ≤ 1, by introducing a ''space cone'' analogous to the well known light cone, but with the spinor helicity technique incorporated through the use of external line factors in a complex gauge. Amplitudes are defined with the choice to identify the two lightlike axes with two of the physical external momenta in accord with the reference momenta choice in the spinor helicity method. The progress along these lines arises by manipulating directly the theory in the Lagrangian and treating the gauge field lines as scalar components. Amplitudes in such theories closer to maximally helicity violating (for spin 1) are also easier to calculate than in previous formulations: For example, in the massless theory even at five-point the helicity structure is such that all amplitudes are within two helicity flips to the maximal case. In these two applications, the known supersymmetry identities become extremely simple, and the closer one is to self-dual helicity configurations the simpler the amplitude calculations are as the couplings in the MHV limit are scalar-like.

Self-dual Yang-Mills field theories may be formulated in a Lorentz covariant fashion through

$$\mathcal{L}=\mathrm{Tr}\, G^{\alpha\beta}F_{\alpha\beta}, \quad F_{\alpha\beta}=\frac{i}{2}\partial_{(\alpha}{}^{\dot\gamma}A_{\beta)\dot\gamma}+A_{\alpha}{}^{\dot\gamma}A_{\beta\dot\gamma}, \quad (1.3)$$

where $G_{\alpha\beta}$ is a multiplier that enforces the self-dual field

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[1]See [14] and [15] for reviews at tree and loop level. Exact sequences of amplitudes include those in [4,16–21].

equations. Different gauge-fixed versions of this action as well as one-field non-Lorentz covariant actions are found in the references (e.g. [10]). Gauge theories may be formulated as a perturbation around this Lorentz covariant action through the addition of the term $G^{\alpha\beta}G_{\alpha\beta}$ followed by integrating over the multipliers. $\mathcal{N}=2$ string theory also admits a description of self-dual and quantum supersymmetric self-dual (the latter possessing a trivial S matrix) description and possesses a gauge theory correspondence with the MHV[2] sector. The perturbative reformulation of gauge theory coupled to matter of various spins near the self-dual point admits improved diagrammatics for calculating amplitudes; in addition, certain chirality properties of the reformulated gauge theory are made manifest. In this work we examine the formulation of massive theories via dualization of perturbed self-dual theories.

The connection between self-duality and quantum field theory in both the classical and quantum regime allows a mapping between the dualized formulations to the $\mathcal{N}=2$ string theories, in both the bosonic and target spacetime supersymmetric versions. The anomalous quantum one-loop amplitudes obtained via sewing the finite number of particles in the $\mathcal{N}=2$ string agree with those of the MHV amplitudes [25]. (The amplitudes are anomalous because they lack a modular invariant integration measure. Otherwise the RNS amplitudes would equal zero at genus one, in either $\mathcal{N}=2$ [31] or $\mathcal{N}=4$ form [32].) The supersymmetric extension analyzed in [25] via superselection sectors generates a trivial S matrix, matching with those of supersymmetric self-dual theories to all loop orders in the covariant version [10]. Perturbation in helicity about the self-dual point allows a direct connection between the $\mathcal{N}=2$ string amplitudes and the self-dual sector of gauge and gravity theory [10].

Many of the ideas behind these works arise in formulating Yang-Mills theory as a perturbation around the self-dual limit, in the sense that the former is described through the first order formulation

$$\mathcal{L}=\mathrm{Tr}\left(-\frac{g^2}{2}G^{\alpha\beta}G_{\alpha\beta}+G^{\alpha\beta}F_{\alpha\beta}+\frac{1}{2}M^2A^{\alpha\dot{\alpha}}A_{\alpha\dot{\alpha}}\right), \tag{1.4}$$

which upon integrating the $A^{\alpha\dot{\alpha}}$ field generates a simultaneously unitarity and renormalizable gauge theory. [The mass term in Eq. (1.4) is obtained via spontaneous symmetry breaking, described in this work.] In the massless limit the lowest order in g term (GF) describes a two-field self-dual Yang-Mills model proposed in [26]. It was perturbatively quantized and solved in [10], generating the color-ordered MHV one-loop S matrix,

$$A_{n;1}(k_1^+,\ldots,k_n^+)=-\frac{i}{48\pi^2}\sum_{1\le i<j<k<l\le n}\frac{\langle ij\rangle[jk]\langle kl\rangle[li]}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}, \tag{1.5}$$

together with a non-vanishing three-point vertex in $d=2+2$ dimensions. Here we listed the leading-in-color contribution for an adjoint vector in the loop with color structure (analogous to attaching Chan-Paton factors to the boundaries of the annulus),

$$\mathrm{Tr}\,T^{\sigma_1}\ldots T^{\sigma_n}, \tag{1.6}$$

where the individual T^{σ_j} are the color matrices for the jth external line. (See [27,28] for related work and [29] for formulations based on non-Lorentz covariant actions.) After integrating out the gauge fields $A^{\alpha\dot{\beta}}$ rather than the tensors $G^{\alpha\beta}$ one arrives at a dual formulation of a gauge theory, written as a sigma model in the tensors $G^{\alpha\beta}$ [13], and in the scalars that generated the mass term. (Related work includes the dualized models [30].) The latter theory also may be used to efficiently derive the amplitudes closer to maximally helicity violating (or self-dual); it furthermore has several applications, from its formulation as a non-linear sigma model of Yang-Mills theory. In this work we explore its application as a dual formulation of spontaneously broken gauge theories.

The outline of this work is the following. In Sec. II we present several examples within the dualized Abelian theory (i.e., Stueckelberg model). The dual theory of the Fermi interactions are obtained naturally in our formulation. In Sec. III we examine general reduction associated with non-Abelian models and derive the corresponding dual formulations. In Sec. IV we examine the new Feynman rules of the dualized non-Abelian theories. In Sec. V we derive the analog of color flow but for the Lorentz group, i.e., spin ordering. We give a four-point massive vector amplitude in Sec. VI as an example. In Sec. VII we give the spin ordering in vector notation. Section VIII contains the derivation of cross sections in the dual formulation. In Sec. IX we examine the (Glashow-Salam-Weinberg) electroweak model. In Sec. X we end with a discussion of further relevant work associated with the self-dual and dualized massive vector theories described here.

II. MASSIVE QED

A. Feynman rules

We first briefly review the dualization of an Abelian theory and give an example scattering process. We also describe the line factors associated with the massive states incorporating spinor helicity techniques.

In [13] we considered the dualization of Abelian vector fields. We begin with a Stueckelberg theory

$$\mathcal{L}=\frac{1}{2}F^{\alpha\beta}F_{\alpha\beta}+\frac{M^2}{2}A^{\alpha\dot{\alpha}}A_{\alpha\dot{\alpha}}, \tag{2.1}$$

where $F^{\alpha\beta}$ is the self-dual field strength,

$$F_{\alpha\beta}=\frac{i}{2}\partial_{(\alpha}{}^{\dot{\beta}}A_{\beta)\dot{\beta}}. \tag{2.2}$$

We also include the fermions through the minimal coupling

[2]MHV represents maximally helicity violating amplitudes for the reason that all partons of out-going momentum have the same helicity.

$$\mathcal{L}_\psi = -\bar{\psi}^{\dot\alpha} i\nabla_{\alpha\dot\alpha}\psi^\alpha - \bar{\chi}^{\dot\alpha} i\nabla_{\alpha\dot\alpha}\chi^\alpha + m(\psi^\alpha\chi_\alpha + \bar{\psi}^{\dot\alpha}\bar{\chi}_{\dot\alpha}). \tag{2.3}$$

For completeness, we will also discuss the free, neutral analogue,

$$\mathcal{L}_\psi = -\frac{1}{2}\bar{\psi}^{\dot\alpha} i\partial_{\alpha\dot\alpha}\psi^\alpha + \frac{1}{2}m(\psi^\alpha\psi_\alpha + \bar{\psi}^{\dot\alpha}\bar{\psi}_{\dot\alpha}) \tag{2.4}$$

for purposes of comparing external line factors. (Of course, in the non-Abelian case this action can be coupled in real representations.) We may express the Lagrangian in Eq. (2.1) in first order form with the addition of the fields $G^{\alpha\beta}$,

$$\mathcal{L} = -\frac{1}{2}G^{\alpha\beta}G_{\alpha\beta} + G^{\alpha\beta}F_{\alpha\beta} + \frac{M^2}{2}A^{\alpha\dot\alpha}A_{\alpha\dot\alpha}. \tag{2.5}$$

Now the action is quadratic in the original gauge field $A^{\alpha\dot\alpha}$, and we may eliminate A in exchange for G (after integrating by parts the ∂A term in the self-dual field strength). Furthermore, we may also integrate out all of the barred components of the fermions as the action is also quadratic in these fields.

In doing so, we obtain the dualized theory,

$$\begin{aligned}\mathcal{L}_{\rm red} = & -\frac{M^2}{2}G^{\alpha\beta}G_{\alpha\beta} - \psi^\alpha(\Box - m^2)\chi_\alpha - \frac{1}{2}\,\frac{1}{M^2-\psi^\alpha\chi_\alpha}\\ & \times\left(\psi^\beta\partial_{\alpha\dot\alpha}\chi_\beta + \frac{1}{2}\partial^\beta{}_{\dot\alpha}[\psi_{(\alpha}\chi_{\beta)} + 2MG_{\alpha\beta}]\right)\\ & \times\left(\psi^\rho\partial^{\alpha\dot\alpha}\chi_\rho + \frac{1}{2}\partial^{\rho\dot\alpha}[\psi^{(\alpha}\chi_{\rho)} + 2MG^\alpha{}_\rho]\right).\end{aligned} \tag{2.6}$$

We shall drop the Jacobians from the integration, which vanish in dimensional reduction (regularization). We further rescaled the fields $\psi\to\sqrt{2m}\psi$, $\chi\to\sqrt{2m}\chi$ and $G_{\alpha\beta}\to MG_{\alpha\beta}$ to simplify the coefficients.[3] Similar reductions may be performed in the Abelian Higgs model. The reduced theory in Eq. (2.6) for the massive particles is simplified in that there is no gamma matrix algebra and only one type of spin index occurs labeling the particle content for both the fermions and gauge bosons. The propagators are of second order form and the dimensions of the *rescaled* fields are $[G]=[\psi]=[\chi]=1$.

The Feynman rules from the dualized theory in Eq. (2.6) contain the propagator for the vectors $G^{\alpha\beta}$ and for the fermion and are both second order. Expanding the inverse of $1-(e/m)^2\psi^\alpha\chi_\alpha$ produces an infinite number of terms. We list the two vertices contributing to the ψ^2G^2 process which entails expanding the action in Eq. (2.6) to second order in the fermions:

$$\begin{aligned}\mathcal{L}_{\rm red} = & \frac{1}{2}G^{\alpha\beta}(\Box - M^2)G_{\alpha\beta} - \psi^\alpha(\Box - m^2)\chi_\alpha\\ & -\frac{1}{M}\psi^\beta\partial_{\alpha\dot\alpha}\chi_\beta\partial^{\rho\dot\alpha}G^\alpha{}_\rho + \frac{1}{2M}G^{\alpha\beta}\Box(\psi_{(\alpha}\chi_{\beta)})\\ & + \frac{1}{2M^2}\psi^\alpha\chi_\alpha(\partial^\beta{}_{\dot\alpha}G_{\alpha\beta})(\partial^{\rho\dot\alpha}G^\alpha{}_\rho) - \frac{1}{2M^2}(\psi^\beta\partial_{\alpha\dot\alpha}\chi_\beta)\\ & \times(\psi^\rho\partial^{\alpha\dot\alpha}\chi_\rho) - \frac{1}{2M^2}\psi^\beta\partial_{\alpha\dot\alpha}\chi_\beta\partial^{\rho\dot\alpha}[\psi^{(\alpha}\chi_{\rho)}]\\ & + \frac{1}{8M^2}\psi^{(\alpha}\chi^{\beta)}\Box(\psi_{(\alpha}\chi_{\beta)}).\end{aligned} \tag{2.7}$$

The propagator for $G^{\alpha\beta}$ is given by

$$\Delta^{\alpha\beta,\gamma\rho}(k) = \frac{1}{k^2+M^2}[C^{\alpha\gamma}C^{\beta\rho} + C^{\alpha\rho}C^{\beta\gamma}]. \tag{2.8}$$

The three-point vertex found from the expansion in Eq. (2.7) is

$$\begin{aligned}&\langle\psi_\alpha(k_1)\chi_\beta(k_2)G_{\mu\nu}(k_3)\rangle\\ &\quad = -\frac{1}{M}\left(C_{\alpha\beta}(k_2k_3)_{\mu\nu} + \frac{1}{2}k_3^2C_{\beta(\mu}C_{\nu)\alpha}\right),\end{aligned} \tag{2.9}$$

in which the second term is also symmetric in α,β, together with the ψ^2G^2 one,

$$\begin{aligned}&\langle\psi_\alpha(k_1)\chi_\beta(k_2)G_{\mu_1\nu_1}(k_3)G^{\mu_2\nu_2}(k_4)\rangle\\ &\quad = -\frac{1}{4M^2}C_{\alpha\beta}(k_3k_4)_{(\mu_1}{}^{(\mu_2}C_{\nu_1)}{}^{\nu_2)},\end{aligned} \tag{2.10}$$

where we have defined the shorthand notation

$$(kp)^{\alpha\beta} = k^{\alpha\dot\alpha}p^\beta{}_{\dot\alpha}. \tag{2.11}$$

Note that only the *undotted* indices appear explicitly in the Feynman rules. The four-point vertex corresponding to $\psi^2\chi^2$ is in k-space,

$$\begin{aligned}&\langle\chi_{\alpha_1}(k_1)\psi_{\beta_1}(k_2)\chi_{\alpha_2}(k_3)\psi_{\beta_2}(k_4)\rangle\\ &\quad = -\frac{1}{2M^2}\Big[k_1\cdot k_3\,C_{\alpha_1[\beta_1}C_{\beta_2]\alpha_2} + (k_1,k_3+k_4)_{\alpha_2[\beta_2}C_{\beta_1]\alpha_1}\\ &\qquad - \frac{1}{4}(k_1+k_2)^2C_{\beta_1(\beta_2}C_{\alpha_2)\alpha_1}\Big].\end{aligned} \tag{2.12}$$

Unlike the undualized theory, there are explicit four-fermion vertices appearing in the rewriting of the original massive QED.

B. External line factors

We next specify the line factors for the massive fermions and vectors through the use of spinor helicity techniques

[3]We denote the mass of fermions with lower case m and the mass of vectors with upper case M in this work.

[33]. Our conventions are as follows: In the *massless* case, Weyl spinors of momentum k in $d=3+1$ are labeled according to $\psi^\alpha(k)=k^\alpha$ and the conjugate $\overline{\chi}^{\dot\alpha}(k)=k^{\dot\alpha}$. (In $d=2+2$ they are independent.) Each massless momentum $k^{\alpha\dot\alpha}$ is associated with a twistor as $k^{\alpha\dot\alpha}=k^\alpha k^{\dot\alpha}$ for positive energy, or $-k^\alpha k^{\dot\alpha}$ for negative. These invariants satisfy $s_{ij}=(k_i+k_j)^2=-\langle ij\rangle[ji]$ in terms of the spinor products $\langle ij\rangle$ of the undotted twistors and $[ij]$ of the dotted.

The *massive* theory in Eq. (2.3) generates the field equation $k_{\alpha\dot\alpha}\psi^\alpha+m\chi_{\dot\alpha}=0$, and its complex conjugate, that possesses two solutions at momentum k for the fermion. In the complex case there are two independent solutions at momentum $-k$, while in the real case ($\chi=\psi$) they are related. The solutions may be specified in terms of *two* Weyl spinors of momentum $k_{(+)}$ and $k_{(-)}$ such that the momentum

$$k^{\alpha\dot\alpha}=k_{(+)}^{\alpha\dot\alpha}+k_{(-)}^{\alpha\dot\alpha}=k_{(+)}^{\alpha}k_{(+)}^{\dot\alpha}+k_{(-)}^{\alpha}k_{(-)}^{\dot\alpha},$$

$$m^2=-k^2=\langle +-\rangle[-+] \tag{2.13}$$

for positive energy (and $k=-k_{(+)}-k_{(-)}$ for negative energy). The "spin vector" $S^{\alpha\dot\alpha}$, satisfying

$$S^2=1, \quad k\cdot S=0 \tag{2.14}$$

whose spatial part defines the axis with respect to which $k_{(\pm)}^\beta$ describe states of $s_z=\pm\frac{1}{2}$, is then

$$mS^{\alpha\dot\alpha}=k_{(+)}^{\alpha\dot\alpha}-k_{(-)}^{\alpha\dot\alpha}. \tag{2.15}$$

Since there are two solutions for the two-component spinor, the choice of basis is arbitrary: The simplest choice is obviously to choose a basis proportional to $k_{(\pm)}^\beta$,

$$\psi_{(\pm)}^\beta=\mu k_{(\pm)}^\beta \Rightarrow \psi_{(+)}^\alpha\psi_{(-)}^\beta-\psi_{(-)}^\alpha\psi_{(+)}^\beta=-\mu^2\langle +-\rangle C^{\alpha\beta} \tag{2.16}$$

using the anti-symmetrization $u_\alpha v_\beta-u_\beta v_\alpha=-u^\gamma v_\gamma C_{\alpha\beta}$.

Since we use 8 (real) components of $k_{(\pm)}^\beta$ to describe 4 components of momentum, we have some freedom for restrictions. A convenient restriction is

$$\langle +-\rangle=[-+]=m \tag{2.17}$$

consistent with Eq. (2.13). This normalization eliminates a phase, however, within the inner products.

Various normalizations of μ can be convenient. For example, the orthonormal basis is

$$\mu=\frac{1}{\sqrt{\langle +-\rangle}}\Rightarrow \psi_{(\pm)}^\beta=\frac{k_{(\pm)}^\beta}{\sqrt{\langle +-\rangle}},$$

$$\psi_{(+)}^\alpha\psi_{(-)}^\beta-\psi_{(-)}^\alpha\psi_{(+)}^\beta=-C^{\alpha\beta}. \tag{2.18}$$

However, for the above restriction on $\langle +-\rangle$, or in general since $|\langle +-\rangle|=m$, this choice is inconvenient for considering massless limits. An alternative, used in conjunction with Eq. (2.17), is to drop the normalization factor:

$$\mu=1\Rightarrow \psi_{(\pm)}^\beta=k_{(\pm)}^\beta, \quad \psi_{(+)}^\alpha\psi_{(-)}^\beta-\psi_{(-)}^\alpha\psi_{(+)}^\beta=-mC^{\alpha\beta}. \tag{2.19}$$

In fact, these are the familiar choices $\overline{u}u=1$ or m for Dirac spinor line factors.

In the orthonormal case, the matrix $\psi_{(\pm)}^\beta$ is just the Lorentz transformation from the rest frame to an arbitrary frame: explicitly,

$$k^{\alpha\dot\alpha}=\psi_{(\beta)}^\alpha(m\,\delta^{\beta\gamma})\overline{\psi}_{(\gamma)}^{\dot\alpha}. \tag{2.20}$$

The little group is SU(2) acting on the "(α)" indices, which leaves the momentum $k^{\alpha\dot\alpha}$ invariant. (Similar remarks apply when applied to the original fermionic actions.) This SU(2) invariance is just the freedom to change basis. For the neutral ("Majorana") fermion, we note that the field equation $k_{\alpha\dot\alpha}\psi^\alpha(k)+m\overline{\psi}_{\dot\alpha}(-k)=0$ relates $\psi_{(+)}$ for negative energy to $k_{(-)}$, etc.:

$$\psi_{(\alpha)}^\alpha(-k_{(+)}-k_{(-)})=\epsilon_{(\alpha)(\beta)}\psi_{(\beta)}^\alpha(k_{(+)}+k_{(-)}). \tag{2.21}$$

Previous to [33], a more complicated choice of external line factors was used [6]. Both can be found by projecting with $\gamma\cdot k+m$ on a massless solution (for just $k_{(2)}$), but the newer choice projects on a Weyl spinor while the older one projects on a Majorana spinor, producing twice as many terms. The two are related by an SU(2) little group transformation: explicitly,

$$\psi_{(\alpha)}'^{\gamma}=\Lambda_{(\alpha)}{}^{(\beta)}\psi_{(\beta)}^\gamma,$$

$$\Lambda_{(\alpha)}{}^{(\beta)}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -[+-]/m\\ -\langle +-\rangle/m & 1\end{pmatrix}. \tag{2.22}$$

This construction generalizes directly to arbitrary spin, since the use of this basis reduces any field equation to the rest frame. The result is always simple for the case of only undotted indices because there are no constraints (other than the Klein-Gordon equation) to satisfy. In contrast, the usual representations with mixed indices always require transversality conditions.

For example, the external line factors $L^{\alpha\beta}$ for the massive vector particles have three components and represent the three independent polarizations of the massive vector field. Boosting from the rest frame to non-trivial momenta associated with each external line immediately yields

$$L_{(\alpha),(\beta)}^{\gamma\delta}=\psi_{(\alpha)}^{(\gamma}\psi_{(\beta)}^{\delta)}, \tag{2.23}$$

with the same normalization as in the spinor case. If we convert the (α) SU(2) indices to 3-vector notation with Pauli σ-matrices for $s_z=\pm 1,0$,

$$\sigma_{+,(\alpha)}{}^{(\beta)}=\begin{pmatrix}0&1\\0&0\end{pmatrix}, \quad \sigma_{-,(\alpha)}{}^{(\beta)}=\begin{pmatrix}0&0\\1&0\end{pmatrix},$$

$$\sigma_{0,(\alpha)}{}^{(\beta)}=\frac{1}{\sqrt{2}}\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \tag{2.24}$$

we have, in the orthonormal basis,

$$L_+^{\alpha\beta} = -i\,\frac{k^\alpha_{(+)}k^\beta_{(+)}}{\langle +-\rangle}, \quad L_-^{\alpha\beta} = i\,\frac{k^\alpha_{(-)}k^\beta_{(-)}}{\langle +-\rangle}, \tag{2.25}$$

$$L_0^{\alpha\beta} = -\frac{i}{\sqrt{2}\langle +-\rangle}\left(k^\alpha_{(+)}k^\beta_{(-)} + k^\alpha_{(-)}k^\beta_{(+)}\right). \tag{2.26}$$

These polarizations satisfy the identities,

$$L_+\cdot L_- = 1, \quad L_0\cdot L_0 = 1, \quad L_+\cdot L_0 = L_-\cdot L_0 = 0, \tag{2.27}$$

where $u\cdot v = u^{\alpha\beta}v_{\alpha\beta}$. The orthogonality relation is

$$\sum_{\lambda=\pm,0} L^\lambda_{\alpha\beta}L^{-\lambda}_{\gamma\delta} = -\frac{1}{2}C_{\gamma(\alpha}C_{\beta)\delta}. \tag{2.28}$$

These relations are useful for cross-section calculations with and without explicit reference momenta choices for the external massive vectors.

The three physical polarization vectors of the massive gauge field $A^{\alpha\dot{\alpha}}$ satisfy

$$k\cdot\varepsilon^\lambda = 0, \quad \varepsilon^\lambda(M,k)\cdot\varepsilon^{-\lambda}(M,k) = 1 \tag{2.29}$$

and

$$\sum_{\lambda=\pm,0}\varepsilon^\lambda_{\alpha\dot{\alpha}}(M,k)\varepsilon^{\star,\lambda}_{\beta\dot{\beta}}(M,k) = C_{\alpha\beta}C_{\dot{\alpha}\dot{\beta}} + \frac{k_{\alpha\dot{\alpha}}k_{\beta\dot{\beta}}}{2M^2}. \tag{2.30}$$

The set of three polarizations satisfying Eq. (2.30) are then also found by boosting with $\psi^\beta_{(\alpha)}$:

$$\varepsilon_+^{\alpha\dot{\alpha}} = \frac{k^\alpha_{(+)}k^{\dot{\alpha}}_{(-)}}{M}, \quad \varepsilon_-^{\alpha\dot{\alpha}} = \frac{k^\alpha_{(-)}k^{\dot{\alpha}}_{(+)}}{M},$$

$$\varepsilon_0^{\alpha\dot{\alpha}} = \frac{1}{\sqrt{2}M}\left(k^\alpha_{(+)}k^{\dot{\alpha}}_{(+)} - k^\alpha_{(-)}k^{\dot{\alpha}}_{(-)}\right). \tag{2.31}$$

These results agree with those found by applying the relation $A^{\alpha\dot{\alpha}} = (i/M)\partial^{\beta\dot{\alpha}}G_\beta{}^\alpha$ to the L's.

The use of these line factors together with our rules represents a simplification over the standard Feynman rules using the polarization vectors $\varepsilon_{\alpha\dot{\alpha}}$ chosen for the massive vector lines. In the latter case the form of the four-component polarization vector is considerably more complicated than the line factors above. Use of these line factors is similar to the use of spinor helicity for the massless vector bosons, in which the two polarization vectors are represented as bispinors,

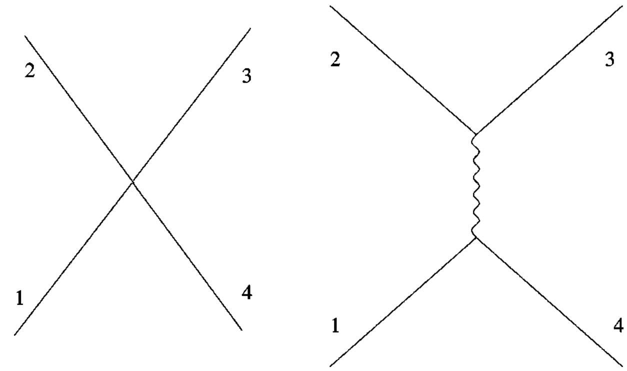

FIG. 1. The two contributions to Möller scattering, $\psi(k_1)\psi(k_2)\rightarrow\chi(k_3)\chi(k_4)$.

$$\varepsilon^+_{\alpha\dot{\alpha}}(k;q) = \frac{q_\alpha k_{\dot{\alpha}}}{q^\beta k_\beta}, \quad \varepsilon^-_{\alpha\dot{\alpha}}(k;q) = -\frac{k_\alpha q_{\dot{\alpha}}}{q^{\dot{\beta}}k_{\dot{\beta}}}. \tag{2.32}$$

Different choices of q in these representations lead to a shift

$$\varepsilon^\pm_{\alpha\dot{\alpha}}(k;q_1) - \varepsilon^\pm_{\alpha\dot{\alpha}}(k;q_2) = f(k;q_1,q_2)k_{\alpha\dot{\alpha}}, \tag{2.33}$$

and the difference is zero on-shell as the longitudinal component decouples. The adaptation of the spinor helicity formalism suitable to describing the polarization vectors of the massive vector bosons follows as in the massive vector case. Our approach generalizates this formalism to the dualized theory and the ambiguity is reflected in the decoupling of the transversal part of the massive gauge field.

C. Scattering

To illustrate the use of the Feynman rules we compute in the dualized theory in Eq. (2.6) the Mǿller and Compton scattering processes, i.e. $\psi\psi\rightarrow\chi\chi$ and $\psi G\rightarrow\chi G$.

In Mǿller scattering we have the two diagrams shown in Fig. 1: (a) an intermediate massive photon line between two fermionic couplings and the permutation, and (b) the four-point fermionic vertex. Labeling spin and momentum, we denote the initial states as $\psi_{\alpha_1}(k_1)$ and $\psi_{\beta_1}(k_2)$ and final states as $\chi_{\alpha_2}(k_3)$ and $\chi_{\beta_2}(k_4)$. The diagram (a) is

$$A_{\alpha_1\beta_1,\alpha_2\beta_2}(k_1,k_2;k_3,k_4) = -\frac{1}{M^2}\left[C_{\beta_1\alpha_2}(k_3,k_4-k_1)_{\mu\nu} + (k_1-k_4)^2C_{\alpha_2\mu}C_{\nu\beta_2}\right]\frac{1}{(k_1-k_4)^2+M^2}\left[C^{\mu\rho}C^{\nu\sigma}+C^{\mu\sigma}C^{\nu\rho}\right]$$
$$\times\left[C_{\alpha_1\beta_2}(k_4,k_1-k_4)_{\rho\sigma} + (k_1-k_4)^2C_{\beta_2\rho}C_{\sigma\alpha_1}\right] \tag{2.34}$$

where the latter terms in the first and second lines are symmetric due to the propagating vector in the intermediate state. The expanded expression is

$$A_{\alpha_1\beta_1,\alpha_2\beta_2}(k_1,k_2;k_3,k_4)=-\frac{1}{M^2}\frac{1}{(k_1-k_4)^2+M^2}\{C_{\beta_1\alpha_2}C_{\alpha_1\beta_2}\mathrm{Tr}[(k_4,k_1)(k_3,k_4)-(k_4-k_1,k_3)(k_4,k_1)]$$
$$-(k_1-k_4)^2C_{\beta_1\alpha_2}[(k_3,k_4-k_1)_{\beta_2\alpha_1}+(k_3,k_4-k_1)_{\alpha_1\beta_2}]$$
$$-(k_1-k_4)^2C_{\alpha_1\beta_2}[(k_4,k_1)_{\alpha_2\beta_1}+(k_4,k_1)_{\beta_1,a_2}]-(k_1-k_4)^4[C_{\alpha_2\beta_2}C_{\alpha_1\beta_1}+C_{\alpha_2\alpha_1}C_{\beta_2\beta_1}]\}, \tag{2.35}$$

and the trace represents the contraction $\mathrm{Tr}(ab)=a^{\mu\nu}b_{\nu\mu}$. The second diagram (b) follows from the vertex in Eq. (2.12), and the amplitude is the sum of the two.

Next we compare with the four fermion Fermi interaction theory [34]. This is obtained by integrating out the massive vector, in the Möller process or by inspection of the dual Lagrangian in Eq. (2.7). The dual theory has the nice feature that the intermediate vector exchange is an order higher in the fermion derivatives (at large values of M) than the $\psi^2\chi^2$ vertex interaction that is obtained directly in the dual theory in Eq. (2.6) (the dual theory is more natural in this regard with the V-A theory). The dual Fermi interactions are modeled by

$$\mathcal{L}_{\mathrm{Fermi}}=-\psi^{\alpha}(\Box-m^2)\chi_{\alpha}-\frac{1}{2M^2}(\psi^{\beta}\partial_{\alpha\dot{\alpha}}\chi_{\beta})(\psi^{\rho}\partial^{\alpha\dot{\alpha}}\chi_{\rho})$$
$$-\frac{1}{2M^2}\psi^{\beta}\partial_{\alpha\dot{\alpha}}\chi_{\beta}\partial^{\rho\dot{\alpha}}[\psi^{(\alpha}\chi_{\rho)}]$$
$$+\frac{1}{8M^2}\psi^{(\alpha}\chi^{\beta)}\Box(\psi_{(\alpha}\chi_{\beta)}). \tag{2.36}$$

The form in Eq. (2.36) is in a second order fermionic form; we can undualize it to write it in first order form via

$$\mathcal{L}_{\mathrm{Fermi}}=-\overline{\psi}^{\dot{\alpha}}i\partial_{\alpha\dot{\alpha}}\psi^{\alpha}-\overline{\chi}^{\dot{\alpha}}i\partial_{\alpha\dot{\alpha}}\chi^{\alpha}+m(\psi^{\alpha}\chi_{\alpha}+\overline{\psi}^{\dot{\alpha}}\overline{\chi}_{\dot{\alpha}})$$
$$+\frac{1}{M^2}\psi^{\alpha}\overline{\psi}^{\dot{\alpha}}\chi^{\beta}\overline{\chi}^{\dot{\beta}}\mathcal{O}_{\alpha\dot{\alpha},\beta\dot{\beta}}. \tag{2.37}$$

Integrating half of the fermionic components in Eq. (2.37) generates the theory in Eq. (2.36) after scaling the fermionic fields by $\psi\to m\psi$, $\chi\to m\chi$ and keeping terms up to the quadratic order in derivatives. The matrix $\mathcal{O}_{\alpha\dot{\alpha},\beta\dot{\beta}}$ is determined in the process. Decoupling the undualized massive vector in the Möller scattering induces the matrix $\mathcal{O}_{\alpha\dot{\alpha},\beta\dot{\beta}}=C_{\alpha\beta}C_{\dot{\alpha}\dot{\beta}}$.

For Compton scattering $\psi G\to\chi G$ there are three diagrams: (1) two with an intermediate photon line via the vertices in Eq. (2.9); (2) one from the four-point vertex in Eq. (2.10). We denote the momenta of the fermions by k_1 and k_2 and those of the photons p_1 and p_2. The indices of the fermions are α_1 and α_2 while those of the photons are μ_j,ν_j. The first diagram gives

$$A_{1,\alpha_1\alpha_2\mu_1\nu_1\mu_2\nu_2}=\left(\frac{1}{M^2}\right)\Big[-C_{\alpha_1\alpha_2}(k_1+p_1,p_1)_{\mu_1\nu_1}(k_2p_2)_{\mu_2\nu_2}$$
$$-\frac{M^2}{2}C_{\alpha_2(\mu_1}C_{\nu_1)\alpha_1}(k_2p_2)_{\mu_2\nu_2}$$
$$+\frac{M^2}{2}(k_1+p_1,p_1)_{\mu_1\nu_1}C_{\alpha_2(\mu_2}C_{\nu_2)\alpha_1}$$
$$+\frac{M^4}{4}C_{\alpha_2(\mu_2}C_{\nu_2)(\mu_1}C_{\nu_1)\alpha_1}\Big]\frac{1}{(k_1+p_1)^2+m^2} \tag{2.38}$$

together with the crossed diagram by symmetrizing with respect to the two photon legs,

$$A_{2,\alpha_1\alpha_2\mu_1\nu_1\mu_2\nu_2}=\left(\frac{1}{M^2}\right)\Big[-C_{\alpha_1\alpha_2}(k_1+p_2,p_2)_{\mu_2\nu_2}(k_2p_1)_{\mu_1\nu_1}$$
$$-\frac{M^2}{2}C_{\alpha_2(\mu_2}C_{\nu_2)\alpha_1}(k_2p_1)_{\mu_1\nu_1}$$
$$+\frac{M^2}{2}(k_1+p_2,p_2)_{\mu_2\nu_2}C_{\alpha_2(\mu_1}C_{\nu_1)\alpha_1}$$
$$+\frac{M^4}{4}C_{\alpha_2(\mu_1}C_{\nu_1)(\mu_2}C_{\nu_2)\alpha_1}\Big]\frac{1}{(k_1+p_2)^2+m^2}. \tag{2.39}$$

The four-point vertex for $\psi\chi G^2$ is identical to that in Eq. (2.10) and generates the final diagram to this process.

Squaring the amplitude to obtain the cross section, via summing over final states and averaging over initial ones, involves the identities and Eq. (2.28) for the external fermion and vector line factors, respectively. We complete the amplitude calculations for the Compton process in the remainder of this section, taking the fermionic matter to be massless. In the massless limit the line factors associated with the second order fermions [22],

$$\epsilon^{+}_{\alpha}(k)=k_{\alpha},\quad \epsilon^{-}_{\alpha}(k)=\frac{q_{\alpha}}{\langle qk\rangle} \tag{2.40}$$

are used in the calculation. The reference momenta and first line factor choices are

$$\psi_\alpha(k_1)=k_{1+,\alpha}, \quad \chi_\alpha(k_2)=\frac{k_{1+,\alpha}}{\langle 1+2+\rangle} \tag{2.41}$$

and

$$L_{1,\mu\nu}(p_1)=-i\frac{k_{1+,\mu}k_{1+,\nu}}{\langle 1+1-\rangle}, \quad p_1=k_{1+}+k_{1-} \tag{2.42}$$

$$L_{2,\mu\nu}(p_2)=i\frac{k_{2-,\mu}k_{2-,\nu}}{\langle 2+2-\rangle}, \quad p_2=k_{2+}+k_{2-}, \tag{2.43}$$

corresponding to the scattering process $\psi G^+\to\chi G^-$. With these choices all terms from the first diagram and third diagram are immediately equal to zero; furthermore we have the normalizations $\langle j+j-\rangle=[j-j+]=M$. Only the second term in the second diagram is non-vanishing and it generates

$$A(+,-)=\frac{\langle 1+2-\rangle^2[2+1-]}{\langle 2+2-\rangle}\frac{1}{(k_{1+}+p_2)^2}. \tag{2.44}$$

The opposite helicity configuration for the gauge bosons, parametrized by

$$L_{1,\mu\nu}(p_1)=i\frac{k_{1-,\mu}k_{1-,\nu}}{\langle 1+1-\rangle}, \quad p_1=k_{1+}+k_{1-} \tag{2.45}$$

$$L_{2,\mu\nu}(p_2)=-i\frac{k_{2+,\mu}k_{2+,\nu}}{\langle 2+2-\rangle}, \quad p_2=k_{2+}+k_{2-}, \tag{2.46}$$

receives a non-vanishing contribution from the fourth term in the first diagram and the first term in the second diagram equals zero. The sum of the contributing terms equals

$$A(-,+)=\frac{M}{(k_{1+}+p_1)^2}\langle 1-2+\rangle + \frac{1}{(k_{1-}+p_2)^2}\bigg[(k_{1+}+k_{2+})^2\frac{\langle 1-1+\rangle}{M} + M([1+2-]+\langle 1-2+\rangle)\bigg], \tag{2.47}$$

in terms of the reference momenta. The two processes $\psi G^\pm\to\chi G^\pm$ are trivially zero after contractions. Squaring the sum of these amplitudes and taking the high-energy limit, $M\to 0$, generates

$$\sum |M|^2\to\frac{16}{M^2}\frac{(p_1\cdot k_{2+})^2}{k_{1+}\cdot p_2}. \tag{2.48}$$

Normalizing by the line factor associated with the fermion $k_{1+}\cdot k_{2+}=(k_{1+}+k_{2-})\cdot k_{2+}-M^2\to p_2\cdot k_{2+}$ gives the known result

$$\sum |\tilde{M}|^2\to\frac{16}{M^2}p_2\cdot k_{1+}p_2\cdot k_{2+}. \tag{2.49}$$

III. NON-ABELIAN DUALIZATIONS

A. Vectors

In this section we shall consider a general spontaneously broken non-Abelian gauge theory [12]; the scalar and fermion matter content is taken to be generic. The dualized form to this theory is obtained by expressing the original minimally coupled theory in first order form using a Lagrange multipler field strength $G_{\alpha\beta}$:

$$\mathcal{L}=\mathrm{Tr}-\frac{g^2}{2}G^{\alpha\beta}G_{\alpha\beta}+G^{\alpha\beta}F_{\alpha\beta} +\nabla^{\alpha\dot\alpha}\phi^\dagger\nabla_{\alpha\dot\alpha}\phi-\overline{\psi}^{\dot\alpha}i\nabla_{\alpha\dot\alpha}\psi^\alpha+V(\phi). \tag{3.1}$$

Integrating out the dualized gauge fields $G^{\alpha\beta}$ gives back the usual field theory formulation. However, this first order form is now quadratic in the gauge field $A_{\alpha\dot\beta}$; we shall rather functionally integrate out $A^{\alpha\dot\beta}$ [13] and arrive at a theory defined in terms of $G_{\alpha\beta}$.

Integrating out the gauge connection for those vectors acquiring a mass through spontaneous symmetry breaking gives rise to the *T*-dualized Lagrangian examined in [13]. Prior to gauge fixing we obtain the theory after integrating out $A_{\alpha\dot\alpha}$,

$$\mathcal{L}^d=\mathrm{Tr}-\frac{1}{2}G^{\alpha\beta}G_{\alpha\beta}+V_{\alpha,a}{}^{\dot\gamma}[X^{-1}]^{\alpha\beta}_{ab}V_{\beta\dot\gamma,b}-\overline{\psi}^{\dot\alpha,a}i\partial_{\alpha\dot\alpha}\psi^\alpha_a +\partial^{\alpha\dot\alpha}\phi^{*,a}\partial_{\alpha\dot\alpha}\phi_a+V(\phi). \tag{3.2}$$

The remaining massless vectors not Higgsed by the scalar interactions and in Eq. (3.2) not indexed by *a*, give the standard (undualized) contribution to the Lagrangian:

$$\mathcal{L}^u=\mathrm{Tr}\frac{1}{2}F^{\alpha\beta}F_{\alpha\beta}+(\nabla^{\alpha\dot\alpha}\phi)^\dagger\nabla_{\alpha\dot\alpha}\phi-\overline{\psi}^{\dot\alpha}i\nabla_{\alpha\dot\alpha}\psi^\alpha. \tag{3.3}$$

The full theory in Eq. (3.1) is $\mathcal{L}_d+\mathcal{L}_u$. The gauge-fixing may be performed before the integration of the gauge field; in unitary gauge this would eliminate one component of the complex scalar. We will do so in the next section relevant to the dual of the electroweak model. In the above, the vector *V* and matrix *X* in Eq. (3.2) are found to be

$$V_a^{\alpha\dot\alpha}=\partial_\gamma{}^{\dot\alpha}G_a^{\alpha\gamma}+i\overline{\psi}^{\dot\alpha}T_a^\psi\psi^\alpha+\phi^\dagger T_a^\phi(\partial^{\alpha\dot\alpha}\phi)-(\partial^{\alpha\dot\alpha}\phi^\dagger)T_a^\phi\phi, \tag{3.4}$$

and

$$X_{ab}{}^{\alpha\beta}=if_{ab}{}^cG_c{}^{\alpha\beta}+\phi^\dagger\{T_a^\phi,T_b^\phi\}\phi\, C^{\alpha\beta}. \tag{3.5}$$

We need to expand the form in Eq. (3.2) about the broken phase in order to generate the Feynman rules, of which there are an infinite number arising from the Taylor expansion about the vacuum value of the scalar field. We write the expansion as $\phi=\tilde\phi+v$, and of course, the lowest order term in Eq. (3.2) is kept by keeping only the quadratic in v term in the inverse of the matrix $X_{ab}^{\alpha\beta}$, which generates the mass term for the vector bosons. These rules will be derived in the following sections.

B. Spinors

Our aim is to eliminate to a large extent the dotted $\dot{\alpha}$ index from appearing in the gauge theory coupled to matter of various spin content. We will see that this allows for a ''color ordering'' for internal spin structures which leads to a further minimization of amplitudes into smaller gauge invariant subsets; the minimization of amplitudes into subsets is advantageous because calculations of smaller gauge-invariant subsets of amplitudes is simpler and there are relations between these subsets (as in the case of subleading in color contributions to partial amplitudes: these subleading trace structure partial amplitudes $A_{n;m}$ can be obtained from $A_{n;1}$). This is analogous to the effect that spinor helicity has in minimizing the redundancy of the gauge invariant interactions and is similar to the color ordered breakup of amplitudes.

We demonstrate here then the dualization of the non-Abelian model in the previous section combined with the reduced fermion Lagrangian derived in [11]. Recall that the fermion reduction is derived by integrating out half of the fermionic components contained in the Lagrangian in Eq. (10).

Upon using the field equation for $\bar{\psi}$, we eliminate this field and obtain the fermionic contribution to the gauge theory in Eq. (1.2); the $\Box$ is gauge covariantized, i.e. $2\Box = \nabla^{\alpha\dot{\alpha}}\nabla_{\alpha\dot{\alpha}} = (\partial^{\alpha\dot{\alpha}} + iA^{\alpha\dot{\alpha}})(\partial_{\alpha\dot{\alpha}} + iA_{\alpha\dot{\alpha}})$, and generates trilinear and quartic couplings. The fully dualized matter theory is found through Eq. (3.1), but together with the fermion contribution in Eq. (1.2) (which is quadratic in $A_{\alpha\dot{\alpha}}$). In the following we combine this integrating out procedure with the dualization of the vector fields.

To generate the dualized system of the non-Abelian gauge theory coupled to fermions we introduce as before a first order form through

$$\mathcal{L}_g = \mathrm{Tr} - \frac{1}{2} G^{\alpha\beta} G_{\alpha\beta} + G^{\alpha\beta} F_{\alpha\beta}. \tag{3.6}$$

We proceed by integrating out the massive vector fields as in the previous section, after including the Lagrangians in Eqs. (3.1) and (3.2). The general form of the dual Lagrangian taking into account the reduced fermion contribution in Eq. (1.2) is then (after integrating out $A^{\alpha\dot{\alpha}}$),

$$\mathcal{L}^d = \mathrm{Tr}\left(-\frac{1}{2} G^{\alpha\beta} G_{\alpha\beta} \right) + V_{\alpha,a}{}^{\dot{\gamma}} [X^{-1}]_{ab}^{\alpha\beta} V_{\beta\dot{\gamma},b} + \partial^{\alpha\dot{\alpha}} \psi^{\beta,a} \partial_{\alpha\dot{\alpha}} \psi_{a,\beta} + \partial^{\alpha\dot{\alpha}} \phi^{*,a} \partial_{\alpha\dot{\alpha}} \phi_a + V(\phi). \tag{3.7}$$

The vector $V_a^{\alpha\dot{\alpha}}$ is given by

$$V_a^{\alpha\dot{\alpha}} = \partial_\gamma{}^{\dot{\alpha}} (G_a^{\alpha\gamma} + \psi^\alpha T_a^\psi \psi^\gamma) + \psi^\gamma T_a^\psi \overleftrightarrow{\partial}^{\alpha\dot{\alpha}} \psi_\gamma + \phi^\dagger T_a^\phi \overleftrightarrow{\partial}^{\alpha\dot{\alpha}} \phi, \tag{3.8}$$

and the matrix $X_{ab}{}^{\alpha\beta}$ is

$$X_{ab}{}^{\alpha\beta} = i f_{ab}{}^c G_c{}^{\alpha\beta} + (\phi^\dagger \{T_a^\phi, T_b^\phi\} \phi + \psi^\gamma \{T_a^\psi, T_b^\psi\} \psi_\gamma) C^{\alpha\beta}. \tag{3.9}$$

Note that the resulting action is complex. We have also left the representation content of the matter under the gauge group arbitrary (T_a^ϕ, T_a^ψ); we shall consider specific representations in the following subsections.

The form (3.7) of the dualized theory may be found by comparison with the first order Lagrangian in Eq. (3.1); the two couplings in Eq. (1.2), the scalar-like box and the self-dual tensor $F_{\alpha\beta}$, couple as the $G^{\alpha\beta}F_{\alpha\beta}$ term and as the scalar $\phi\Box\phi$ ones. Note that in distinction to the result in Eq. (1.2), the fermions enter into the denominator of the dualized theory through the matrix $X_{ab}^{\alpha\beta}$; the entire contribution for X^{-1} must be expanded about the vacuum value leading to an infinite number of interaction terms (but a finite number at given order in perturbation theory). In this reduced dualized theory we have chirally minimized the theory in the sense that all of the fields are labeled by only one type of spin index, i.e. $G^{\alpha\beta}$, ψ^α, and ϕ. A single (for fermions) and double (for gauge fields) line formulation naturally follows associated with the contractions of the α-type index.

IV. FEYNMAN RULES

In this section we will derive the Feynman rules to the reduced dualized theory of the previous section. (In Sec. VI, we will apply these to the example of the four-point vector amplitude.) We also make explicit the double line representation for the Feynman graphs in accord with the spin ordering.

To find the couplings to fourth order in the gauge fields, we need to expand the inverse to the matrix $X_{ij}^{\alpha\beta}$ to second order about its background values of the scalar fields. The matrix X has the form

$$X_{ij}^{\alpha\beta} = G_k^{\alpha\beta} M_{ij}^k + Q_{ij} C^{\alpha\beta}. \tag{4.1}$$

The first term in Eq. (4.1) is symmetric in (α,β) and anti-symmetric in (i,j), and the second one is anti-symmetric in (α,β) and symmetric in (i,j).

The perturbative form of the inverse to the matrix X is obtained from the expansion

$$\left[\frac{1}{X}\right]_\alpha^{ij,\beta} = \frac{1}{G_{(m)} M^{(m)} + Q\delta} = \frac{1}{Q}\left[\frac{1}{\delta + Q^{-1} G_{(m)} M^{(m)}}\right], \tag{4.2}$$

around the mass matrix $Q_{ij} = \phi^\dagger \{T_i^\phi, T_j^\phi\} \phi$. Expanding $X = \langle X\rangle + \triangle X$ is not direct because there is a field dependent coefficient multiplying the $\langle X\rangle$ term. A first order expansion generates Feynman rules to third order in the G-fields,

$$\left[\frac{1}{X}\right]^{ij,\beta}_{\alpha}=[Q^{-1}_{ik}][\delta_\alpha{}^\beta\delta_{kj}-Q^{-1}_{kl}\,G_{(a)\alpha}{}^{\beta}M^{(a)}_{lj}+\cdots]$$
$$=\delta_\alpha{}^\beta Q^{-1}_{ij}+Q^{-1}_{ik}Q^{-1}_{kl}G_\alpha{}^{\beta(a)}M^{(a)}_{lj}+\cdots. \quad (4.3)$$

Higher order terms to the inverse matrix are obtained by performing the Taylor expansion and contain more powers of the matrix $G^{\alpha\beta}_{(k)}M^{(k)}_{ij}$. Including the fermion contributions, for example, gives the lowest order expansion:

$$\left[\frac{1}{X}\right]^{ij,\beta}_{\alpha}=\delta_\alpha{}^\beta Q^{-1}_{ij}+Q^{-1}_{ik}Q^{-1}_{kl}[G_\alpha{}^{\beta(a)}M^{(a)}_{lj}$$
$$+\delta_\alpha{}^\beta\psi^\gamma\{T^\psi_l,T^\psi_j\}\psi_\gamma]+\cdots, \quad (4.4)$$

with the Q_{ij} matrix the same form, i.e. $Q_{ij}=\phi^\dagger\{T^\phi_i,T^\phi_j\}\phi$. Although we may write a general form for the perturbative inverse, we shall not do so here as it is not relevant to the following discussion. These expansions suffice to generate the quartic couplings of the gauge fields.

We now describe how the group theory color ordering combines with the spin ordering. The mass matrix for the scalars in the fundamental representation has the particularly simple form in an $SU(2)$ gauge theory,

$$Q_{ab}=\phi^\dagger\{T^\phi_a,T^\phi_b\}\phi=\frac{1}{2}\,\delta_{ab}\,\phi^\dagger\phi. \quad (4.5)$$

This form of the mass matrix Q_{ij} simplifies the derivation of the Feynman rules in this section because its inverse is diagonal. On the other hand, the general fundamental representation of $SU(N)$ satisfies the identity

$$\{T^\phi_a,T^\phi_b\}=\frac{1}{N}\,\delta_{ab}+d_{abc}T^c, \quad (4.6)$$

and the mass matrix is only symmetric. Along the directions $d_{abc}v^\dagger T^c v=0$, the inverse is also quite simple. The inverse for general representations incorrectly prohibits factoring out of the group theory via color ordering because we need to determine the appropriate color flow along the propagators. The color flow with the metric generated for the propagators above in the general case is provided in a general representation of the matter content.

The fourth order in $G_{\alpha\beta}$ Feynman rules require a second order expansion of $1/X^{ab}_{\alpha\beta}$. We first define the inverse matrix to the background values of the mass matrix by

$$I_{ac}\,v^\dagger\{T^\phi_c,T^\phi_b\}v=\delta_{ab}\,. \quad (4.7)$$

We shall drop the couplings to scalars; the theory is

$$\mathcal{L}_r=[\partial_\gamma{}^{\dot\alpha}(G^{\alpha\gamma}_a+\psi^\alpha T^\psi_a\psi^\gamma)+\psi^\beta T^\psi_a\overleftrightarrow{\partial}^{\alpha\dot\alpha}\psi_\beta]\left[\frac{1}{X}\right]^{\alpha\beta}_{ab}$$
$$\times[\partial_{\rho\dot\alpha}(G^{\beta\rho}_b+\psi^\beta T^\psi_b\psi^\rho)+\psi^\mu T^\psi_b\overleftrightarrow{\partial}^\beta{}_{\dot\alpha}\psi_\mu], \quad (4.8)$$

where to second order we have

$$\left[\frac{1}{X}\right]^{\alpha\beta}_{ab}=\frac{1}{(v^\dagger v)}C^{\alpha\beta}I_{ab}+\frac{1}{(v^\dagger v)^2}I_{ac}I_{cd}(G^{\alpha\beta}_nM^{(n)}_{db}$$
$$+C^{\alpha\beta}\psi^\gamma\{T^\psi_d,T^\psi_b\}\psi_\gamma)+\frac{1}{8(v^\dagger v)^3}I_{ac}I_{cd}(G^{\alpha\delta}_nM^{(n)}_{de}$$
$$+C^{\alpha\delta}\psi^\gamma\{T^\psi_d,T^\psi_e\}\psi_\gamma)I_{ef}(G_{\delta n}{}^\beta M^{(n)}_{fb}$$
$$+\delta_\delta{}^\beta\psi^\gamma\{T^\psi_f,T^\psi_b\}\psi_\gamma). \quad (4.9)$$

The metric I_{ab} is the metric used to contract the color indices arising at the vertices. The net color flow, even for general I_{ab}, may be factored out as in the usual formulation of Yang-Mills theory.

This expansion of the inverse matrix may be thought of as one in the dimension of the operators, or inversely in the counting of the Higgs vacuum value which enters through the background value of the mass matrix. The color ordered Feynman rules are simplest when I_{ab} is diagonal [e.g. $SU(2)$ fundamental].

The vertices in Eq. (4.8) possess two momenta with a string of Kronecker deltas absorbing the indices; this suggests a simple double line representation associated with the α indices to the Feynman rules similar to that of color flow (with Chan-Paton factors). In the diagrams we represent a Kronecker delta symbol connecting the indices at either end by a single line. We have to also represent the product $k_{i\alpha\dot\alpha}k_j^{\beta\dot\alpha}$ from the contraction of two momenta in the vertices; a dot on the line indicates that the matrix $k_{i\alpha\dot\alpha}k_j^{\beta\dot\alpha}$ is used to soak up the indices on the lines *i* and *j*. The general fermion or gauge boson (i.e. $G^{\alpha\beta}$) line possesses these contractions, as can be seen in $V^{\alpha\dot\alpha}_a$ in Eq. (3.8).

V. SPIN ORDERING

We now examine in detail the lower-point vertices and describe the spin ordering in practice. The lowest order expansion generates the propagator in the form

$$\mathcal{L}_{G^2}=\frac{I_{ab}}{2M^2}\,G^{\alpha\beta,a}[\Box-M^2]G^b_{\alpha\beta}, \quad (5.1)$$

with I_{ab} defined in Eq. (4.7). We rescale $G^{\alpha\beta}\to MG^{\alpha\beta}$ here and in the following to simplify the form of the propagator and vertices. In momentum space the color ordered propagator has the two equivalent tensor structures

$$\langle G_\alpha{}^\beta(k)G_\mu{}^\nu(-k)\rangle=\frac{1}{k^2+M^2}[\delta^\nu_\alpha\delta^\beta_\mu+\delta_{\alpha\mu}\delta^{\beta\nu}]$$
$$=\frac{1}{k^2+M^2}[2\,\delta^\nu_\alpha\delta^\beta_\mu-\delta^\beta_\alpha\delta^\nu_\mu]. \quad (5.2)$$

The group theory factor associated with the propagator is contained in the color factor I_{ab}. The forms of the propagators are illustrated within the double line formulation in Fig. 2. In Fig. 2 we have not included the color flow, but represent the lines as contractions of the α,β-type vertices.

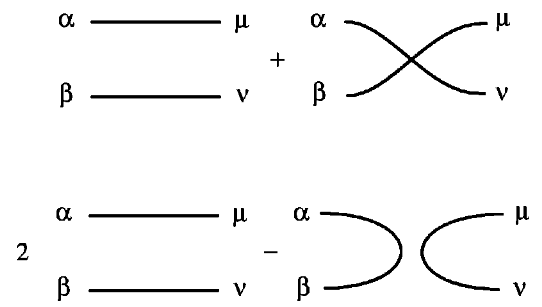

FIG. 2. The two terms contributing to the propagator.

The three- and four-point couplings containing only the G-fields are obtained from the interactions

$$\mathcal{L}_{G^3}=\frac{1}{M}(\partial_{\gamma}{}^{\dot\alpha}G_a^{\alpha\gamma})G_{b,\alpha\beta}(\partial_{\rho\dot\alpha}G_c^{\beta\rho}), \tag{5.3}$$

$$\mathcal{L}_{G^4}=\frac{1}{M^2}(\partial_{\gamma}{}^{\dot\alpha}G_a^{\alpha\gamma})\,G_{b,\alpha}{}^{\beta}G_{c,\beta}{}^{\delta}(\partial_{\rho\dot\alpha}G_{d,\delta}{}^{\rho}), \tag{5.4}$$

where the tensors $I_a{}^dI_d{}^ef^{bc}{}_e$ and $I_{ae}I^{ei}f^b{}_{ig}I_{gh}f_{hb}{}^c$ represent the color structure and have been removed in the above; they are obvious from the graphical interpretation. We shall also use the double line representation to graphically represent the vertices; unlike the propagator, we must use a line with a dot to represent the momenta contraction on the lines.

The three-point vertex found from $\mathcal{L}_{G^3}$ contains the terms

$$\begin{aligned}\langle G_{a_1}^{\alpha_1}{}_{\beta_1}(k_1)G_{a_2}^{\alpha_2}{}_{\beta_2}(k_2)G_{a_3}^{\alpha_3}{}_{\beta_3}(k_3)\rangle &\\ =-\frac{1}{M}[I^{a_1d}I_d{}^ef^{a_2a_3}{}_e(12)^{\alpha_1}{}_{\beta_2}\,\delta_{\beta_3}^{\alpha_2}\delta_{\beta_1}^{\alpha_3} &\\ +I^{a_2d}I_d{}^ef^{a_3a_1}{}_e(23)^{\alpha_2}{}_{\beta_3}\,\delta_{\beta_1}^{\alpha_3}\delta_{\beta_2}^{\alpha_1} &\\ +I^{a_3d}I_d{}^ef^{a_1a_2}{}_e(31)^{\alpha_3}{}_{\beta_1}\,\delta_{\beta_2}^{\alpha_1}\delta_{\beta_3}^{\alpha_2}],\end{aligned} \tag{5.5}$$

where we have explicitly summed over the ordering of the external labels to make the rule Bose symmetric. In addition we have to symmetrize in the pairs of indices (α_1,β_1), (α_2,β_2), and (α_3,β_3); we do not include these terms because the propagator and external line factors explicitly twist these indices and symmetrizes the pairs. From now on we shall drop the color matrices and leave the result color-ordered with the exception of the following four-point example; furthermore we use the notation

$$(ij)^{\alpha\beta}=k_i{}^{\alpha\dot\alpha}k_j^{\beta}{}_{\dot\alpha} \tag{5.6}$$

to define the often recurring matrix from the vertex algebra.

The color-ordered three-point vertex is found by taking the (123) ordering of the group trace structures represented by the Levi-Cevita tensors,

$$f^{a_1a_2a_3}=\mathrm{Tr}([T^{a_1},T^{a_2}]T^{a_3}). \tag{5.7}$$

In eliminating the group theory factors we find the color ordered form of the three-point vertex to be

$$\begin{aligned}V_{123}{}^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3}=\frac{1}{M}[(12)^{\alpha_1}{}_{\beta_2}\,\delta_{\beta_3}^{\alpha_2}\delta_{\beta_1}^{\alpha_3}+(23)^{\alpha_2}{}_{\beta_3}\,\delta_{\beta_1}^{\alpha_3}\delta_{\beta_2}^{\alpha_1} &\\ +(31)^{\alpha_3}{}_{\beta_1}\,\delta_{\beta_2}^{\alpha_1}\delta_{\beta_3}^{\alpha_2}].\end{aligned} \tag{5.8}$$

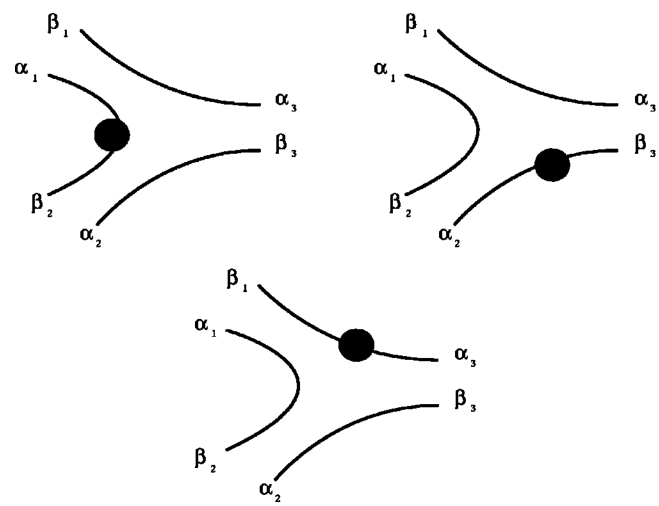

FIG. 3. The three terms of the color-ordered three-point vertex.

This vertex is illustrated graphically in Fig. 3. The dot in Fig. 3 denotes the contraction of the momenta as found in the vertex in Eq. (5.8) from the factors (ij).

The momentum space four-point vertex is derived in a similar fashion. We label all of the lines with the pairs of indices (α_i,β_i); its Bose symmetric form produces the terms in the case of $SU(2)$,

$$\langle G_{a_1}^{\alpha_1\beta_1}(k_1)G_{a_2}^{\alpha_2\beta_2}(k_2)G_{a_3}^{\alpha_3\beta_3}(k_3)G_{a_4}^{\alpha_4\beta_4}(k_4)\rangle$$

$$\begin{aligned}=\frac{1}{M^2}[&\epsilon^{a_1a_2l}\epsilon_l{}^{a_3a_4}(12)^{\alpha_1}{}_{\beta_2}\,\delta_{\beta_3}^{\alpha_2}\delta_{\beta_4}^{\alpha_3}\delta_{\beta_1}^{\alpha_4}+\epsilon^{a_4a_1l}\epsilon_l{}^{a_3a_2}(41)^{\alpha_4}{}_{\beta_1}\,\delta_{\beta_2}^{\alpha_1}\delta_{\beta_3}^{\alpha_2}\delta_{\beta_4}^{\alpha_3}+\epsilon^{a_2a_3l}\epsilon_l{}^{a_1a_4}(23)^{\alpha}{}_{\beta_3}\,\delta_{\beta_4}^{\alpha_3}\delta_{\beta_1}^{\alpha_4}\delta_{\beta_2}^{\alpha_1}\\ &+\epsilon^{a_3a_4l}\epsilon_l{}^{a_2a_1}(34)^{\alpha_3}{}_{\beta_4}\,\delta_{\beta_1}^{\alpha_4}\delta_{\beta_2}^{\alpha_1}\delta_{\beta_3}^{\alpha_2}+\epsilon^{a_1a_2l}\epsilon_l{}^{a_3a_4}(12)^{\alpha_1}{}_{\beta_2}\,\delta_{\beta_4}^{\alpha_2}\delta_{\beta_3}^{\alpha_4}\delta_{\beta_1}^{\alpha_4}+\epsilon^{a_4a_1l}\epsilon_l{}^{a_2a_3}(41)^{\alpha_4}{}_{\beta_1}\,\delta_{\beta_3}^{\alpha_1}\delta_{\beta_2}^{\alpha_3}\delta_{\beta_4}^{\alpha_2}\\ &+\epsilon^{a_2a_3l}\epsilon_l{}^{a_4a_1}(23)^{\alpha_2}{}_{\beta_3}\,\delta_{\beta_2}^{\alpha_4}\delta_{\beta_4}^{\alpha_1}\delta_{\beta_1}^{\alpha_3}+\epsilon^{a_3a_4l}\epsilon_l{}^{a_1a_2}(34)^{\alpha_3}{}_{\beta_4}\,\delta_{\beta_2}^{\alpha_4}\delta_{\beta_3}^{\alpha_1}\delta_{\beta_1}^{\alpha_2}+\cdots].\end{aligned} \tag{5.9}$$

There are in addition terms not included in the symmetric sum which do not possess a possible ordering $\epsilon^{a_1 a_2 m}\epsilon_m{}^{a_3 a_4}$ or $\epsilon^{a_2 a_3 m}\epsilon_m{}^{a_4 a_1}$ of the group theory factors. (The contractions of the epsilon tensors can be written as products of Kronecker delta functions, but in a color ordering the group theory factorizes directly from the kinematics.) The eight terms above in Eq. (5.9) must also be twisted within the pairs (α_i,β_i) to generate the complete four-point vertex; however, as with the three-point vertex the propagator and external line factors explicitly perform the twisting of these indices. We may strip the color from the above vertex by using

$$\epsilon^{a_1 a_2 l}\epsilon_l{}^{a_3 a_4}=\mathrm{Tr}[T^{a_1},T^{a_2}][T^{a_3},T^{a_4}], \tag{5.10}$$

and collecting all contributions to the trace structure with the ordering (1234).

In general, for $SU(N)$, we need the metric I_{ab} to join the trace generators at each external line. Doing so gives the color-ordered form the same as in the $SU(2)$ example,

$$\begin{aligned}V_{1234}{}^{\alpha_1\alpha_2\alpha_3\alpha_4}_{\beta_1\beta_2\beta_3\beta_4}=\frac{1}{M^2}[&(12)^{\alpha_1}{}_{\beta_2}\,\delta^{\alpha_2}_{\beta_3}\delta^{\alpha_3}_{\beta_4}\delta^{\alpha_4}_{\beta_1}\\&-(41)^{\alpha_4}{}_{\beta_1}\,\delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2}_{\beta_3}\delta^{\alpha_3}_{\beta_4}-(23)^{\alpha_2}{}_{\beta_3}\,\delta^{\alpha_3}_{\beta_4}\delta^{\alpha_4}_{\beta_1}\delta^{\alpha_1}_{\beta_2}\\&-(34)^{\alpha_3}{}_{\beta_4}\,\delta^{\alpha_4}_{\beta_1}\delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2}_{\beta_3}+(12)^{\alpha_1}{}_{\beta_2}\,\delta^{\alpha_2}_{\beta_4}\delta^{\alpha_4}_{\beta_3}\delta^{\alpha_3}_{\beta_1}\\&+(41)^{\alpha_4}{}_{\beta_1}\,\delta^{\alpha_1}_{\beta_3}\delta^{\alpha_3}_{\beta_2}\delta^{\alpha_2}_{\beta_4}+(23)^{\alpha_2}{}_{\beta_3}\,\delta^{\alpha_4}_{\beta_2}\delta^{\alpha_1}_{\beta_4}\delta^{\alpha_3}_{\beta_1}\\&+(34)^{\alpha_3}{}_{\beta_4}\,\delta^{\alpha_4}_{\beta_2}\delta^{\alpha_1}_{\beta_3}\delta^{\alpha_2}_{\beta_1}].\end{aligned} \tag{5.11}$$

This vertex is represented in Fig. 4.

We end our discussion of the Feynman rules with the couplings of the gauge bosons with the fermions; the simplest example is the coupling between two G-fields and two fermions. From the four-point interaction,

$$\mathcal{L}_{GG\psi\psi}=\frac{1}{2M^2}(\partial_\gamma{}^{\dot\alpha}G_a^{\alpha\gamma})(\partial_{\rho\dot\alpha}G^{\rho}_{b,\alpha})\,\psi^{\delta}\{T^{\psi}_a,T^{\psi}_b\}\psi_\delta, \tag{5.12}$$

we obtain the Feynman rule

$$\begin{aligned}&\langle G_a^{\alpha_1}{}_{\beta_1}(k_1)G_b^{\alpha_2}{}_{\beta_2}(k_2)\psi_i^{\alpha_3}(k_3)\psi_j^{\alpha_4}(k_4)\rangle\\&\quad=\frac{1}{2M^2}k_1^{\alpha_1\dot\alpha}k_{2,\beta_1\dot\alpha}\,\delta^{\alpha_2}_{\beta_2}C^{\alpha_3\alpha_4}\{T^{\psi}_a,T^{\psi}_b\}_{ij}.\end{aligned} \tag{5.13}$$

Another example vertex at the five-point level is obtained from the term

$$\mathcal{L}_{G^3\psi^2}=\frac{1}{M^4}\epsilon^{abd}(\partial_\gamma{}^{\dot\alpha}G_a^{\alpha\gamma})G_b^{\alpha\beta}(\partial_{\rho\dot\alpha}G_c^{\beta\rho})\psi^{\delta}\{T^{\psi}_d,T^{\psi}_c\}\psi_\delta. \tag{5.14}$$

There are many other vertices describing the couplings of fermions and $G^{\alpha\beta}$ fields found from the third order Lagrangian in Eq. (4.8) which are easily obtained, and we do not list. A straightforward expansion of the Lagrangian in Eq. (3.2) generates these terms through the expansion of the inverse. An important feature of all of the vertices to any order is that they contain two powers of momentum contracted in the fashion $k_i^{\alpha\dot\alpha}k_j{}^{\gamma}{}_{\dot\alpha}$.

VI. FOUR-POINT MASSIVE VECTOR AMPLITUDE

In this section we give a derivation of a four-point tree amplitude between massive vector bosons. We first introduce the spin trace, defined by

$$\mathrm{Tr}[AB\cdots C]=A_\alpha{}^{\beta}B_\beta{}^{\gamma}\ldots C_\rho{}^{\alpha}. \tag{6.1}$$

There are three topologies of Feynman diagrams which contribute to the four-point tree amplitude. The external line factors for the massive fields, $L_i^{\alpha\beta}$, are described in Sec. II and have a simple form in the rest frame. These matrix line factors must be traced over in the calculations relevant to the four-point massive vector scattering. The first set of Feynman diagrams contains naively a pole in the s_{12} channel; the diagrams give the result

$$\begin{aligned}A_I(k_1,k_2,k_3,k_4)=\frac{1}{M^2}\frac{1}{s_{12}+M^2}\mathrm{Tr}([&L_1(12)L_2-(1+2,1)\\&\times L_1L_2-L_1L_2(2,1+2)][L_3(34)L_4\\&-L_3L_4(4,3+4)-(3+4,3)L_3L_4]).\end{aligned} \tag{6.2}$$

Its counterpart with a cupped propagator leads to a multiple trace structure,

$$\begin{aligned}A_I^t(k_1,k_2,k_3,k_4)=\frac{1}{M^2}\frac{1}{s_{12}+M^2}\mathrm{Tr}[&L_1(12)L_2-(1+2,1)\\&\times L_1L_2-L_1L_2(2,1+2)]\mathrm{Tr}[L_3(34)L_4\\&-L_3L_4(4,3+4)-(3+4,3)L_3L_4].\end{aligned} \tag{6.3}$$

The second class of diagrams contains a pole in the s_{23} channel; their contribution is

$$\begin{aligned}A_{II}(k_1,k_2,k_3,k_4)=\frac{1}{M^2}\frac{1}{s_{23}+M^2}\mathrm{Tr}([&L_1(14)L_4-(1+4,1)\\&\times L_1L_4-L_1L_4(4,1+4)][L_3(32)L_2\\&-L_3L_2(2,2+3)-(2+3,3)L_3L_2]).\end{aligned} \tag{6.4}$$

Their twisted counterparts are

$$A_{II}^t(k_1,k_2,k_3,k_4)=\frac{1}{M^2}\frac{1}{s_{23}+M^2}\mathrm{Tr}[L_1(14)L_4-(1+4,1)\times L_1L_4-L_1L_4(4,1+4)]\mathrm{Tr}[L_3(32)L_2-L_3L_2(2,2+3)-(2+3,3)L_3L_2]. \quad (6.5)$$

The remaining contribution to the four-point tree amplitude comes from the four-point vertex, which contains twelve terms:

$$A_{III}=\frac{1}{M^2}\mathrm{Tr}[L_1(12)L_2L_3L_4+L_1L_2L_3L_4(41)+L_1L_2L_3(34)L_4+L_1L_2(23)L_3L_4]+\frac{1}{M^2}\mathrm{Tr}[L_1(12)L_2L_4L_3+L_1L_3L_2L_4(41)+L_2(23)L_3L_1L_4+L_1L_3(34)L_4L_2]. \quad (6.6)$$

The sum of these three combinations generates the spin-ordered (and color-ordered) massive four-point vector amplitude. The line factors associated with the external states in Eq. (2.26) are required for the complete covariant result.

We complete this section with the calculation of the massive analog of the maximally helicity violating process, i.e. scattering of the massive vector bosons possessing the same helicity $G^{\pm}$. We write the momenta of the external vectors in terms of null momenta as $p_j=k_j+q$, with q the same for all of the distinct bosons. The line factors for the out-going G^- states are

$$L^-_{j,\alpha\beta}=i\frac{q_\alpha q_\beta}{\langle qj\rangle}. \quad (6.7)$$

Inspecting the contributions in Eqs. (6.2) to (6.6) we see that all of the terms contain a contraction of $q^\alpha q_\alpha$ and are individually zero. The tree-level helicity process for this configuration of the massive gauge bosons is equal to zero (and similarly at n-point).

VII. VECTORIAL SPIN ORDERING

In the vector theory in the previous section we examined the Feynman rules and amplitudes within the double line representation of the spin algebra. Because $G^{\alpha\beta}$ has three components and transforms like a 3-vector, it is suitable to describe the interactions in vector notation, in which the Feynman rules are made simpler. We shall reformulate in this section the interactions in this notation.

We start by noting that a symmetric matrix $M_{\alpha\beta}$ has three components and may be written as a three-vector, M^i. We need to reexpress the trace formula in the previous section into three-vector notation. Using the identity

$$2A_\alpha{}^\beta B_{\beta\gamma}=A_{(\alpha|}{}^\beta B_{\beta|\gamma)}+A_{[\alpha|}{}^\beta B_{\beta|\gamma]}, \quad (7.1)$$

we find a version of the trace formula

$$\mathrm{Tr}(ABCD)=A\cdot BC\cdot D-A\cdot CB\cdot D+A\cdot DB\cdot C, \quad (7.2)$$

where we may translate into three vector notation with $A\cdot B=A^iB^i$ and

$$A^{\alpha\beta}B_{\alpha\beta}\to\frac{1}{2}A^iB_i. \quad (7.3)$$

The Feynman rules in three-vector notation are obtained by translating the previously derived ones.

The three-point color-ordered vertex in three-vector notation translates into, after contracting with the polarization vectors $L_{\pm,0}$,

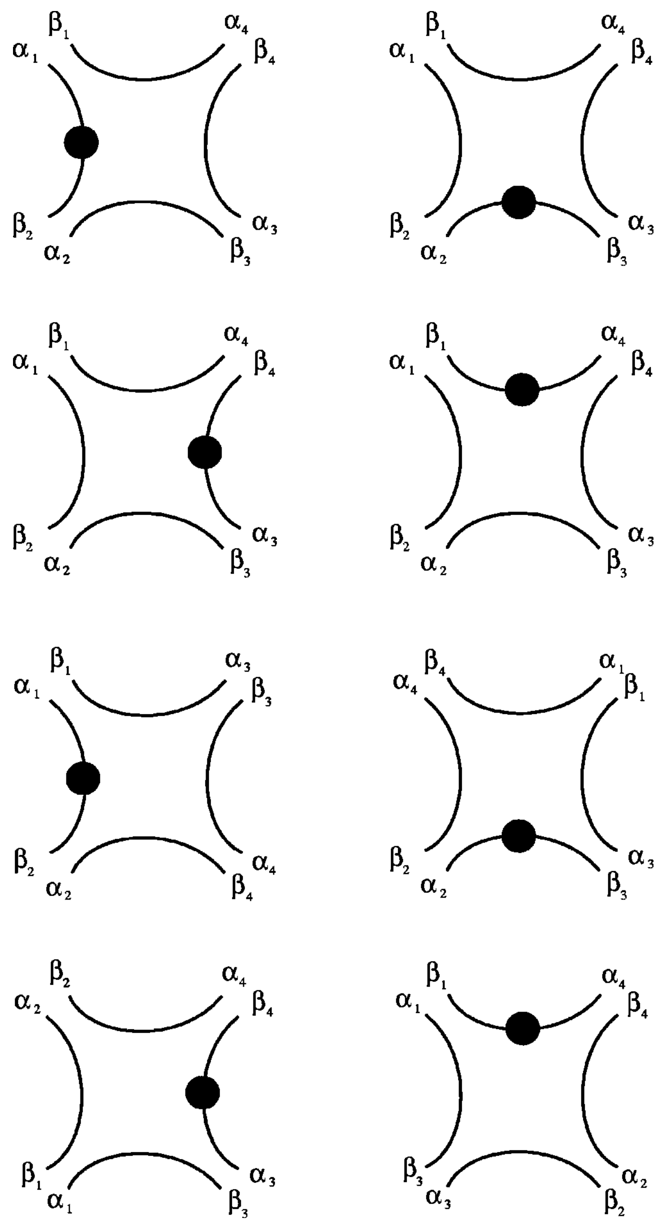

FIG. 4. Spin ordered diagrams of the color-ordered four-point vertex.

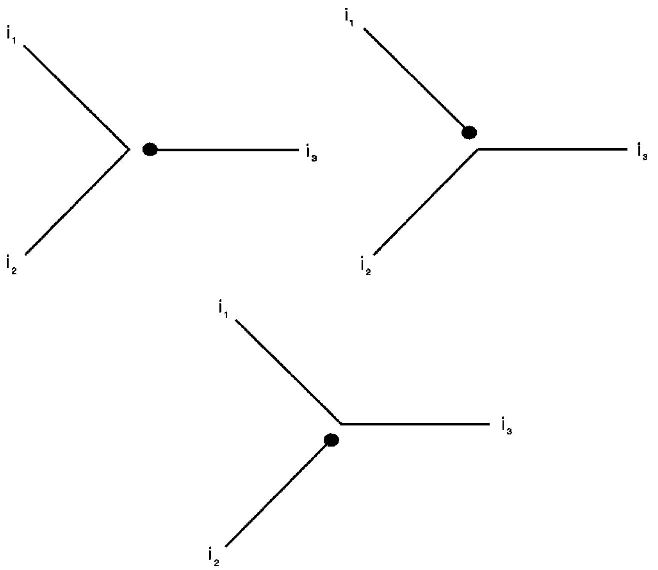

FIG. 5. The three terms of the three-point vertex in the single line representation.

$$V_3 = -\frac{1}{M}[L_1 \cdot L_2 L_3 \cdot (12) + L_1 \cdot L_3 L_2 \cdot (13) + L_2 \cdot L_3 L_1 \cdot (23)] \tag{7.4}$$

or in uncontracted form,

$$V_3^{i_1 i_2 i_3} = -\frac{1}{M}[\delta^{i_1 i_2}(12)^{i_3} + \delta^{i_1 i_3}(13)^{i_2} + \delta^{i_2 i_3}(23)^{i_1}] \tag{7.5}$$

where we have suppressed the helicity dependence in the line factors in Eq. (7.5). This vertex has a particularly simple form when compared with the usual formulations. The vertex is illustrated graphically using the single line representation in Fig. 5; the dot denotes the contraction (ij) of momenta from the two adjacent lines.

We may find a similar form for the color-ordered four-point vertex; this vertex becomes in three-vector notation, after contracting with the external polarizaton vectors,

$$V_4 = -\frac{2}{M^2}[L_1 \cdot L_2 (L_3 \times L_4) \cdot (34) + L_2 \cdot L_3 (L_4 \times L_1) \cdot (41)] - \frac{2}{M^2}[L_3 \cdot L_4 (L_1 \times L_2) \cdot (12) + L_4 \cdot L_1 (L_2 \times L_3) \cdot (23)]. \tag{7.6}$$

The four-point vertex has only four terms and is a reduced version of that appearing in the double-line representation. In an uncontracted form, we have the color-ordered four-point vertex written as

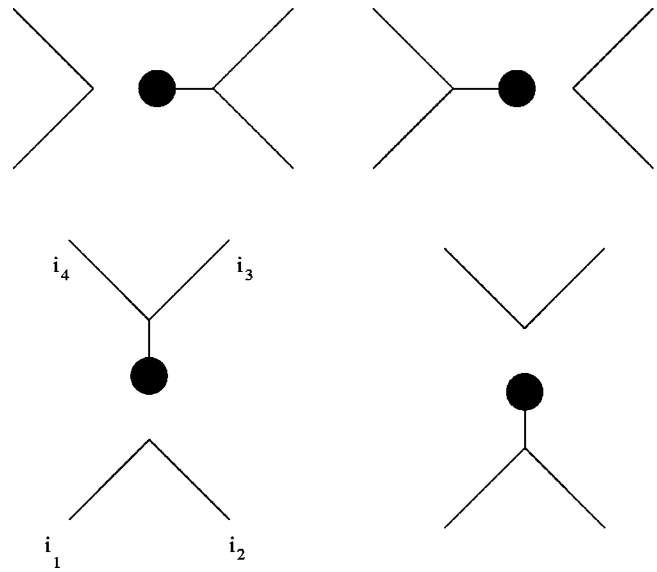

FIG. 6. The four terms contributing to the color-ordered four-point vertex in the single line representation. The dot denotes contraction with the product $\epsilon^{abi}(mn)_i$ of the opposing two lines m and n.

$$V_4^{i_1 i_2 i_3 i_4} = -\frac{2}{M^2}[\delta^{i_1 i_2}\,\epsilon^{i_3 i_4 i}(34)_i + \delta^{i_2 i_3}\,\epsilon^{i_4 i_1 i}(41)_i] - \frac{2}{M^2}[\delta^{i_3 i_4}\,\epsilon^{i_1 i_1 i}(12)_i + \delta^{i_4 i_1}\,\epsilon^{i_2 i_3 i}(23)_i]. \tag{7.7}$$

This vertex is illustrated in the single line representation in Fig. 6.

The forms of these vector indexed vertices are particularly simple and imply a similar simplification of the three- and four-point double line graphs; their form after converting the vector line into a bi-spinor one gives the three and four-point vertices illustrated in Figs. 7 and 8. We have essentially enlarged the vector into a bispinor label; the contraction of the momenta is denoted by the dot on the line. These forms are equivalent to the previous double line representations.

VIII. CROSS SECTIONS

We conclude this section with an explanation on how to obtain the cross sections from the amplitudes derived using the dualized Feynman rules. First, we denote the three independent polarizations in a three-component bi-spinor form, $L^{\lambda}_{\alpha\beta} \rightarrow L^{\gamma\delta}_{\alpha\beta}$. Using this notation, our helicity amplitudes are written as

$$A(k_1,\lambda_1;k_2,\lambda_2, \ldots) \rightarrow A^{\alpha_1\beta_1,\alpha_2\beta_2,\cdots}(k_1,k_2, \ldots). \tag{8.1}$$

There are two ways to make the complex conjugate and the corresponding cross-sections. First, we may use the actual complex conjugation of our amplitude in the form

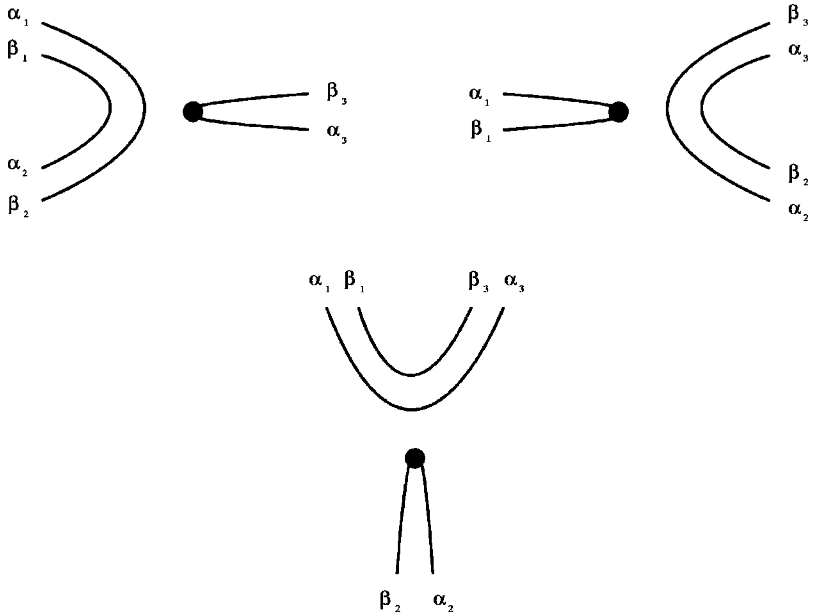

FIG. 7. The alternative double line representation of the three-point vertex.

$$A^*(k_1,\lambda_1;k_2,\lambda_2,\ldots)\to A^{\dot\alpha_1\dot\beta_1,\dot\alpha_2\dot\beta_2,\cdots}(k_1,k_2,\ldots). \tag{8.2}$$

The amplitude squared and summed over the first p polarizations must be contracted with a vector containing both undotted and dotted indices. The only vectors available are the momenta $k_{i,\alpha\dot\alpha}$. The squared amplitude which is summed over the first m polarizations is contracted in the Lorentz covariant manner,

$$\sum_{\text{pol}} |A|^2 = A^{\alpha_1\beta_1\cdots\alpha_n\beta_n}(k_i)A^{\dot\alpha_1\dot\beta_1\cdots\dot\alpha_n\dot\beta_n}(k_i) \times\left[\left(\frac{k_{1,(\dot\alpha_1|(\alpha_1}}{M}\frac{k_{1,\beta_1)|\dot\beta_1)}}{M}\right)\cdots \times\left(\frac{k_{m,(\dot\alpha_m|(\alpha_m}}{M}\frac{k_{m,\beta_m)|\dot\beta_m)}}{M}\right)\right], \tag{8.3}$$

following from the orthogonality over the intermediate spin states $\Sigma_{\lambda=\pm,0}L_\lambda^{\alpha\beta}L_{-\lambda}^{\dot\alpha\dot\beta}$.

In the completely chiral form the dualized Lagrangian defines first our amplitude $A^{\alpha_1\beta_1,\alpha_2\beta_2\cdots}$; the complex conjugate amplitude is found from the action written in terms of $G^{\dot\alpha\dot\beta}$ together with flipping the helicity states from the external line factors. This results in the same form as previously (but with all i's exchanged with $-i$'s). We denote this amplitude $\tilde A^{\alpha_1\beta_1,\alpha_2\beta_2\cdots}(k_1,\lambda_1;k_2,\lambda_2,\ldots)$. Then the cross-section is derived by the orthogonality relation in Eq. (2.28) to join A together with $\tilde A$:

$$\sum_{\text{pol}} |A|^2 = A^{\alpha_1\beta_1\cdots\alpha_n\beta_n}(k_i)\tilde A^{\tilde\alpha_1\tilde\beta_1\cdots\tilde\alpha_n\tilde\beta_n}(k_i) \times[(C_{(\tilde\alpha_1|(\alpha_1}C_{\beta_1)|\tilde\beta_1)})\cdots(C_{(\tilde\alpha_m|(\alpha_m}C_{\beta_m)|\tilde\beta_m)})], \tag{8.4}$$

where the tilde denotes a separate index labeling the same representation as the untilded indices. The latter definition of the cross section is simpler to implement in practice as only one type of index is used and spin ordering is naturally useful.

All final states are summed over in this manner. We may also perform the average over initial states by contracting and summing over their polarizations.

IX. ELECTROWEAK MODEL

A. Dualization

In this section we will consider the well known $SU(2)_L\times U(1)$ electroweak portion of the standard model. The dualized Lagrangian is derived after integrating out as in the previous section the gauge fields for the massive vector bosons from a first order from of gauge theory.

We first review the construction of this model. The Lagrangian for the $SU(2)_L\times U(1)$ theory contains the gauge and scalar sectors,

$$\mathcal{L}_g=\frac{1}{2}F^{\alpha\beta}F_{\alpha\beta}+\frac{1}{2}F^{a,\alpha\beta}F_{a,\alpha\beta}, \tag{9.1}$$

and

$$\mathcal{L}_s=\nabla_{\alpha\dot\alpha}\Phi^\dagger\nabla^{\alpha\dot\alpha}\Phi-\mu^2\Phi^\dagger\Phi+\lambda[\Phi^\dagger\Phi]^2. \tag{9.2}$$

The scalar covariant derivative weighted with the appropriate hypercharge assignments is

$$\nabla_{\alpha\dot\alpha}=\partial_{\alpha\dot\alpha}+ig_2A^a_{\alpha\dot\alpha}\left(\frac{t_a}{2}\right)-\frac{i}{2}g_1A_{\alpha\dot\alpha}, \tag{9.3}$$

where the scalar field is in doublet form,

$$\Phi=\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}. \tag{9.4}$$

Furthermore, we may add doublet fermions through their minimal couplings to the gauge fields, $\mathcal{L}_f = -\bar{\Psi}^{\dot{\alpha}} i \nabla_{\alpha\dot{\alpha}} \Psi^{\alpha}$. The generation of an electron and its neutrino, for example, is

$$\Psi = \begin{pmatrix} \psi_e \\ \psi_{\nu_e} \end{pmatrix}. \tag{9.5}$$

We may couple further generations to the gauge fields.

The standard form of the electroweak model is obtained by going first to a unitary gauge and then making a field redefinition which expresses the massive vector fields $W^{\pm}$ and Z in terms of the massive gauge fields A^a and B. In obtaining the dualized theory there is freedom in making the various steps to obtain the new Lagrangian. Furthermore, we shall use a Stueckelberg mass term for the photon to treat this field on the same footing as the W and Z fields; the amplitudes obtained for the massless photon case may be found by taking the massless limit. (The scalar component of the massive vector must decouple from the S matrix in this limit.) The procedure we adopt in obtaining our formulation of the electroweak model is described by (1) writing a first order form to Eq. (9.1), (2) choose a unitary gauge, (3) integrate out the massive vector bosonic gauge fields. One may interchange steps (1) and (2) in deriving the final form for the Lagrangian.

The first order form to the electroweak model is found by introducing into the Lagrangian in Eq. (9.1) the field strengths $G^A_{\alpha\beta}$ and $G^B_{\alpha\beta}$,

$$\frac{1}{2} F^{a,\alpha\beta} F_{a,\alpha\beta} \Rightarrow -\frac{1}{2} G^{a,\alpha\beta} G_{a\alpha\beta} + G^{a,\alpha\beta} F_{a,\alpha\beta}, \tag{9.6}$$

and

$$\frac{1}{2} F^{\alpha\beta} F_{\alpha\beta} \Rightarrow -\frac{1}{2} G^{\alpha\beta} G_{\alpha\beta} + G^{\alpha\beta} F_{\alpha\beta}. \tag{9.7}$$

The unitary gauge is specified by using the $SU(2)_L$ gauge freedom to choose the gauge $\Phi = (1/\sqrt{2})(\phi + v)I$. We further use the standard basis for the gauge fields $W^{\pm}_{\alpha\dot{\alpha}}$, $Z_{\alpha\dot{\alpha}}$, and photon $A_{\alpha\dot{\alpha}}$ fields:

$$Z_{\alpha\dot{\alpha}} = \frac{g_2 A^3_{\alpha\dot{\alpha}} - g_1 A_{\alpha\dot{\alpha}}}{[g_1^2 + g_2^2]^{1/2}}, \quad B_{\alpha\dot{\alpha}} = \frac{g_1 A^3_{\alpha\dot{\alpha}} + g_2 A_{\alpha\dot{\alpha}}}{[g_1^2 + g_2^2]^{1/2}},$$

$$W_{\alpha\dot{\alpha}} = \frac{1}{\sqrt{2}} [A^1_{\alpha\dot{\alpha}} \pm i A^2_{\alpha\dot{\alpha}}]. \tag{9.8}$$

In the unitary gauge the scalar kinetic term becomes

$$\nabla_{\alpha\dot{\alpha}} \Phi^{\dagger} \nabla^{\alpha\dot{\alpha}} \Phi = \frac{1}{2} \partial_{\alpha\dot{\alpha}} \phi \partial^{\alpha\dot{\alpha}} \phi \tag{9.9}$$

$$+ \frac{g_2^2}{4} (v + \phi)^2 W^+ \cdot W^- + \frac{1}{8} (g_1^2 + g_2^2)(v + \phi)^2 Z \cdot Z. \tag{9.10}$$

The pure Higgs couplings give through the Φ^4 potential we have chosen

$$\mathrm{Tr} \frac{\lambda}{4} (\Phi^2 - v^2)^2 = \frac{1}{2} m_h^2 \phi^2 + \frac{\lambda}{4} (4 v \phi^3 + \phi^4). \tag{9.11}$$

Next we must integrate out the combinations of fields $A^a_{\alpha\dot{\alpha}}$ and $B_{\alpha\dot{\alpha}}$ that correspond to the $W^{\pm}$ and Z field. We shall reexpress the first-order form of the Lagrangian in these gauge field variables.

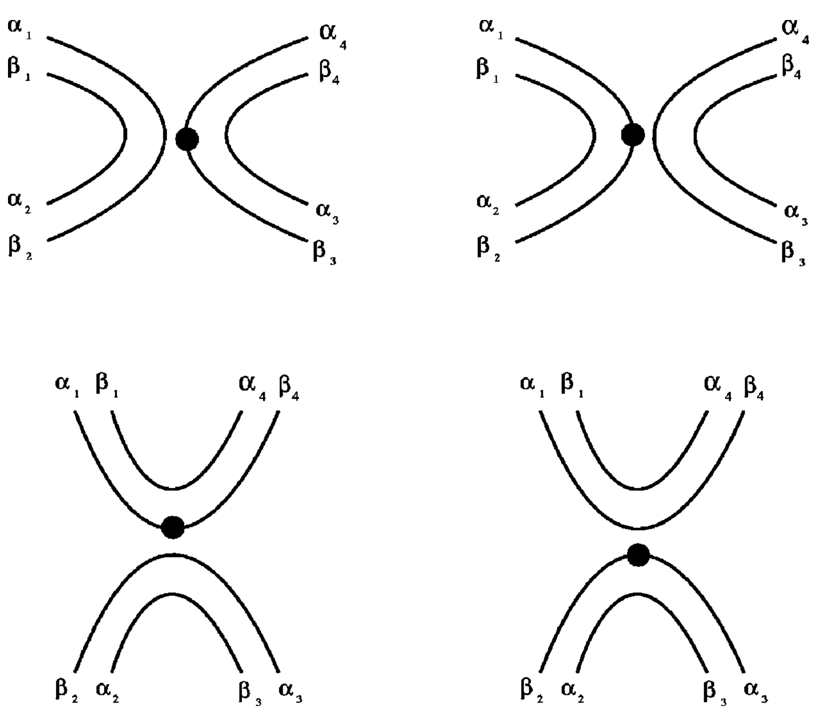

FIG. 8. The alternative double line representation of the four-point vertex.

First we examine the contributions from the dualized field strengths. In terms of the redefined fields we have from the $U(1)$-field contribution,

$$G^{\alpha\beta}F_{\alpha\beta}=i\kappa\, G^{\alpha\beta}\partial_{\alpha\dot{\gamma}}(g_2B-g_1Z)^{\dot{\gamma}}{}_{\beta} \tag{9.12}$$

where $\kappa=1/\sqrt{g_1^2+g_2^2}$. The remaining contributions found from inverting the basis in Eq. (9.8) are expressed as

$$G^{a,\alpha\beta}F^a_{\alpha\beta}=G^{1,\alpha\beta}\left\{\frac{i}{\sqrt{2}}\partial_{\alpha\dot{\gamma}}(W^+ + W^-)^{\dot{\gamma}}{}_{\beta}+\frac{`\kappa}{\sqrt{2}}(g_2Z+g_1B)_{\alpha\dot{\gamma}}(W^+ - W^-)^{\dot{\gamma}}{}_{\beta}\right\}$$
$$+G^{2,\alpha\beta}\left\{\frac{1}{\sqrt{2}}\partial_{\alpha\dot{\gamma}}(W^+ - W^-)^{\dot{\gamma}}{}_{\beta}-\frac{i\kappa}{\sqrt{2}}(g_2Z+g_1B)_{\alpha\dot{\gamma}}(W^+ + W^-)^{\dot{\gamma}}{}_{\beta}\right\}$$
$$+G^{3,\alpha\beta}\left\{\kappa\,\partial_{\alpha\dot{\gamma}}(g_2Z+g_1B)^{\dot{\gamma}}{}_{\beta}-\frac{1}{2}(W^+ + W^-)_{\alpha\dot{\gamma}}(W^+ - W^-)^{\dot{\gamma}}{}_{\beta}\right\}. \tag{9.13}$$

We shall also use a new basis for the G-fields,

$$G^{\alpha\beta}_B=\frac{1}{\sqrt{g_1^2+g_2^2}}(g_1G^{\alpha\beta}_3+g_2G^{\alpha\beta}),\quad G^{\alpha\beta}_Z=\frac{1}{\sqrt{g_1^2+g_2^2}}(g_2G^{\alpha\beta}_3-g_1G^{\alpha\beta}),\quad G^{\alpha\beta}_{W^\pm}=\frac{1}{\sqrt{2}}(G^{\alpha\beta}_1\pm iG^{\alpha\beta}_2). \tag{9.14}$$

Before integrating out the massive fields we write the above in the compact notation which is useful for the Gaussian functional integration; furthermore, we add the mass term for the photon field,

$$G^{a,\alpha\beta}F^a_{\alpha\beta}+G^{\alpha\beta}F_{\alpha\beta}+\frac{g^2}{4}(v+\phi)^2W^+\cdot W^- +\frac{1}{8}(g_1^2+g_2^2)(v+\phi)^2Z\cdot Z+\frac{m^2}{2}B^{\alpha\dot{\alpha}}B_{\alpha\dot{\alpha}}$$
$$=(\partial_{\alpha\dot{\gamma}}\tilde{G}^{i,\alpha}{}_{\beta})\,Z^{j,\beta\dot{\gamma}}+Z^{i}_{\alpha}{}^{\dot{\gamma}}(G^{\alpha\beta}_kM^k_{ij}+Q_{ij}C^{\alpha\beta})\,Z^j_{\beta\dot{\gamma}} \tag{9.15}$$

where $Z=(W^+,W^-,Z,B)$ and $\tilde{G}=(G_{W^+},G_{W^-},G_Z,G_B)$. The matrices M, N, and the vector P are determined from the expansion in Eq. (9.13).

The functional integration over the set of Z_j fields in Eq. (9.15) is Gaussian and may be performed in a straightforward fashion. We obtain

$$\mathcal{L}=V^{\alpha\dot{\gamma}}_i\left[\frac{1}{X}\right]^{ij}_{\alpha\beta}V^{\beta}_{j}{}_{\dot{\gamma}}-\frac{1}{2}G^{\alpha\beta}_ZG_{Z,\alpha\beta}-\frac{1}{2}G^{\alpha\beta}_BG_{B\alpha\beta}-G^{\alpha\beta}_{W^+}G_{W^-,\alpha\beta} \tag{9.16}$$

where

$$V^{\alpha\dot{\gamma}}_i=\partial_{\gamma}{}^{\dot{\gamma}}\tilde{G}^{\gamma\alpha}_i \tag{9.17}$$

and

$$X^{ij}_{\alpha\beta}=G_{k,\alpha\beta}M^k_{ij}+Q_{ij}C_{\alpha\beta}. \tag{9.18}$$

The matrix $G^{\alpha\beta}_kM^k_{ij}$ is

$$G^{\alpha\beta}_kM^k_{ij}=\begin{pmatrix} 0 & \frac{\kappa}{2}(g_2G_Z+g_1G_B)^{\alpha\beta} & -\frac{g_2\kappa}{2}G^{\alpha\beta}_{W^-} & -\frac{1}{2}g_1\kappa G^{\alpha\beta}_{W^-} \\ -\frac{\kappa}{2}(g_2G_Z+g_1G_B)^{\alpha\beta} & 0 & \frac{g_2\kappa}{2}G^{\alpha\beta}_{W^+} & \frac{1}{2}g_1\kappa G^{\alpha\beta}_{W^+} \\ \frac{g_2\kappa}{2}G^{\alpha\beta}_{W^-} & -\frac{g_2\kappa}{2}G^{\alpha\beta}_{W^+} & 0 & 0 \\ \frac{1}{2}g_1\kappa G^{\alpha\beta}_{W^-} & -\frac{1}{2}g_1\kappa G^{\alpha\beta}_{W^+} & 0 & 0 \end{pmatrix}. \tag{9.19}$$

The mass matrix Q_{ij} is similarly obtained from the spontaneously broken contribution in Eq. (9.15) and is

$$Q_{ij}=\frac{1}{2}\begin{pmatrix} 0 & \frac{1}{4}(v+\phi)^2 g_2^2 & 0 & 0 \\ \frac{1}{4}(v+\phi)^2 g_2^2 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4}(v+\phi)^2(g_1^2+g_2^2) & 0 \\ 0 & 0 & 0 & m^2 \end{pmatrix}, \tag{9.20}$$

where the Stueckelberg mass parameter is m^2. The Feynman rules are obtained after perturbatively inverting the matrix X around the mass parameters of the $G_{W^\pm}$, G_Z, and G_B fields. Note that we must add in a mass term for the photon at an intermediate stage to perform the dualization; we take $m=0$ at the end of the S-matrix calculations.

B. Feynman rules

In this section we give the Feynman rules for the dualized electroweak model presented in the previous section. The matrix $X_{ab}^{\alpha\beta}$ has the same form as in Eq. (4.1); the perturbative inverse to first order is

$$\left[\frac{1}{X}\right]_{\alpha\beta}^{ij}=[C_\alpha{}^\gamma Q_{ik}^{-1}][C_{\gamma\beta}\delta_{kj}-(CQ)_{kl}^{-1}{}_\gamma{}^\rho G_{(a)\rho\beta}M_{lj}^{(a)}+\cdots] \tag{9.21}$$

$$=C_{\alpha\beta}Q_{ij}^{-1}+Q_{ik}^{-1}Q_{kl}^{-1}G_{\alpha\beta(a)}M_{lj}^{(a)}+\cdots, \tag{9.22}$$

where the inverse to the matrix Q_{ij} is

$$Q_{ij}^{-1}=2\begin{pmatrix} 0 & \frac{4}{(v+\phi)^2}\frac{1}{g_2^2} & 0 & 0 \\ \frac{4}{(v+\phi)^2}\frac{1}{g_2^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{4}{(v+\phi)^2}\frac{1}{(g_1^2+g_2^2)} & 0 \\ 0 & 0 & 0 & \frac{1}{m^2} \end{pmatrix}. \tag{9.23}$$

This expansion generates couplings to third order in the massive $\tilde{G}_j^{\alpha\beta}$ fields. We may also expand in the scalar field ϕ around the vacuum value v in Eq. (9.23).

Explicitly the $V_i^{\alpha\dot{\alpha}}$ vector is

$$V_j^{\alpha\dot{\gamma}}=\begin{pmatrix} \partial_\gamma{}^{\dot{\gamma}}G_{W^+}^{\gamma\alpha} \\ \partial_\gamma{}^{\dot{\gamma}}G_{W^-}^{\gamma\alpha} \\ \partial_\gamma{}^{\dot{\gamma}}G_Z^{\gamma\alpha} \\ \partial_\gamma{}^{\dot{\gamma}}G_B^{\gamma\alpha} \end{pmatrix}. \tag{9.24}$$

The lowest order term in Eq. (9.22) generates the propagators,

$$\mathcal{L}_0^p=\frac{8}{v^2(g_1^2+g_2^2)}G_{Z,\alpha\beta}(\Box-m_Z^2)G_Z^{\alpha\beta}+\frac{16}{v^2g_2^2}G_{W^+,\alpha\beta}(\Box-m_W^2)G_{W^-}^{\alpha\beta}+\frac{2}{m^2}G_B^{\alpha\beta}(\Box-m_B^2)G_{B,\alpha\beta}, \tag{9.25}$$

where the masses are

$$m_W^2=\frac{v^2g_2^2}{16},\quad m_Z^2=\frac{v^2(g_1^2+g_2^2)}{16},\quad m_B^2=\frac{1}{4}m^2. \tag{9.26}$$

We have used the fact that $\partial_\alpha{}^{\dot{\gamma}}\partial_{\beta\dot{\gamma}}=-C_{\alpha\beta}\Box$.

The interactions to third order in $\tilde{G}_j^{\alpha\beta}$ require the first correction in Eq. (9.22). We need the following matrix form of the first correction:

$$Q_{ik}^{-1}Q_{kl}^{-1}G_{\alpha\beta(a)}M_{lj}^{(a)}=\begin{pmatrix} 0 & \alpha(g_2G_Z+g_1G_B)^{\alpha\beta} & \beta_1 G_{W^-}^{\alpha\beta} & \gamma_1 G_{W^-}^{\alpha\beta} \\ -\alpha(g_2G_Z+g_1G_B)^{\alpha\beta} & 0 & -\beta_1 G_{W^+}^{\alpha\beta} & -\gamma_1 G_{W^-}^{\alpha\beta} \\ \beta_2 G_{W^-}^{\alpha\beta} & -\beta_2 G_{W^+}^{\alpha\beta} & 0 & 0 \\ \gamma_2 G_{W^-}^{\alpha\beta} & -\gamma_2 G_{W^-}^{\alpha\beta} & 0 & 0 \end{pmatrix}, \tag{9.27}$$

where

$$\alpha=\frac{\kappa}{g_2^4}\frac{32}{(v+\phi)^2}, \quad \beta_1=-\frac{\kappa}{g_2^3}\frac{32}{(v+\phi)^2}, \quad \beta_2=g_2\kappa^5\frac{32}{(v+\phi)^2}, \quad \gamma_1=-\frac{1}{2m^4}g_1\kappa, \quad \gamma_2=\frac{g_1\kappa}{g_2^4}\frac{32}{(v+\phi)^2}. \tag{9.28}$$

Since the matrix $Q_{ik}^{-1}Q_{kl}^{-1}$ has only diagonal entries, the matrix $Q_{ik}^{-1}Q_{kl}^{-1}G_{\alpha\beta(a)}M_{lj}^{(a)}$ is very simple. There are several interactions at this order. We have the interactions involving only the G-fields,

$$\begin{aligned}\mathcal{L}_{G^3}=&\alpha_1(g_2G_Z+g_1G_B)_{\alpha\beta}[(\partial_{\rho\dot\gamma}G_{W^-}^{\rho\alpha})(\partial_\gamma{}^{\dot\gamma}G_{W^+}^{\gamma\beta})-(\partial_{\rho\dot\gamma}G_{W^+}^{\rho\alpha})(\partial_\gamma{}^{\dot\gamma}G_{W^-}^{\gamma\beta})]+(\beta_1-\beta_2)(\partial_\gamma{}^{\dot\gamma}G_Z^{\gamma\beta})\\ &\times[(\partial_{\rho\dot\gamma}G_{W^+}^{\rho\alpha})G_{W^-,\alpha\beta}-(\partial_{\rho\dot\gamma}G_{W^-}^{\rho\alpha})G_{W^+,\alpha\beta}]+(\gamma_1-\gamma_2)(\partial_\gamma{}^{\dot\gamma}G_B^{\gamma\beta})[(\partial_{\rho\dot\gamma}G_{W^+}^{\rho\alpha})G_{W^-,\alpha\beta}-(\partial_{\rho\dot\gamma}G_{W^-}^{\rho\alpha})G_{W^+,\alpha\beta}].\end{aligned} \tag{9.29}$$

We end this section with a brief discussion of the next order terms. The expansion of the inverse matrix in Eq. (9.22) to second order yields generically the terms

$$\mathcal{L}_{G^4}=A^{ijkl}(\partial_\gamma{}^{\dot\gamma}G_i^{\gamma\alpha})G_{j,\alpha\rho}G_k{}^\rho{}_\beta(\partial_{\mu\dot\gamma}G_l^{\mu\beta}), \tag{9.30}$$

where $G_i=(G_{W^+},G_{W^-},G_Z,G_B)$, and the coefficients of the interactions and the rules can be determined. The couplings of the $SU(2)_L$ duals has the same form as the vertex in Eq. (5.4).

X. CONCLUSIONS

In this work we have explored the self-dual inspired reformulation of massive vector theories. These manipulations have several advantages: We have a new spin-ordering analogous to color flow and the Feynman rules are suitable (when coupled to matter) to rederiving amplitudes closer to maximally helicity violating ones in an efficient manner.

We have presented the self-dual reformulation for a spontaneously broken gauge theory (and Stuekelberg models as examples) with general matter content. Furthermore, the example of the electroweak model is analyzed. The Lagrangians contain an infinite number of terms: This arises from the reshuffling of the perturbative expansion at arbitrary order. Although the resulting action is more complicated, the Feynman diagrams are simpler, since manipulations normally performed repeatedly at each propagator and vertex in each diagram have been performed once and for all in the corresponding terms in the action. In the massless limit the expansions are formulated around the MHV (or self-dual) configurations and allows for a perturbative analysis in helicity flips at arbitrary n-point.

It would be interesting to demonstrate the use of these rules in practical applications for massive vector processes at higher order. This would entail a calculation (i.e. 6-point) or further work at loop level. As the tensor algebra is reduced and spin-ordered (the diagrams written in smaller gauge invariant sets as well as labeled by only one chiral spinor index), the integral evaluations are the more complicated aspect. In our approach we have reduced the spin dependence of the vertices and interactions. This aspect is the primary complication associated with a direct higher-than-four-point calculation.

There are also interesting aspects to formulating spontaneously broken Yang-Mills theory with gauge group H as a dim(scalars)$+3\times$dim(H) non-linear sigma model. This dual formulation is chiral and written in terms of self-dual field strengths, an analog of holomorphy in $\mathcal{N}=2$ super-Yang-Mills theory. It might be interesting to interpret the isometries of this dual non-linear sigma theory within the conventional formulation of the gauge theory and also the topologically non-trivial gauge configurations and transformations.

Self-dual gauge theory admits an infinite dimensional symmetry structure; as the former is a truncation of gauge theory [10] these conserved currents exist in a limit of the full Yang-Mills theory. The self-dual approach may permit further progress in determining structure in the full gauge theory related to these conserved currents.

ACKNOWLEDGMENTS

The work of G.C. is supported in part by the U.S. Department of Energy, Division of High Energy Physics, Contract W-31-109-ENG-38. The work of W.S. is supported in part by NSF Grant No. PHY 9722101.

APPENDIX: STANDARD ELECTROWEAK MODEL

The bosonic sector of the electroweak model is listed below to facilitate the comparison with our work. We break the bosonic contributions into several terms, and in the following we denote $\underline{a}=\alpha\dot{\alpha}$ and define the field $Y^{\alpha\dot{\alpha}}$ by

$$Y_{\underline{a}}=\cos\theta_w Z_{\underline{a}}+\sin\theta_w B_{\underline{a}}. \tag{A1}$$

The mixing angle is given by $\cos\theta_W = g_2/(g_1^2+g_2^2)^{1/2}$.

The propagators for the gauge fields are found from

$$\mathcal{L}_p=\frac{1}{2}F^{\alpha\beta}F_{\alpha\beta}+\frac{1}{2}F^{Z,\alpha\beta}F^{Z}{}_{\alpha\beta}+F^{W^+,\alpha\beta}F^{W^-}{}_{\alpha\beta} + \frac{m_Z^2}{2}Z\cdot Z+m_W^2 W^+\cdot W^- \tag{A2}$$

where the field strengths are defined in the manner $F^{W^+}_{\alpha\beta} = i\partial_{(\alpha}{}^{\dot{\gamma}}W^+{}_{\beta)\dot{\gamma}}$; the mass parameters are

$$m_w=\frac{1}{2}g_2 v,\quad m_z=\frac{1}{2}(g_1^2+g_2^2)^{1/2}v. \tag{A3}$$

The tri-linear couplings of the gauge fields are found to be

$$\mathcal{L}_t=ig_2(\partial_{\underline{a}}W^+{}_{\underline{b}}-\partial_{\underline{b}}W^+{}_{\underline{a}})W^{-,\underline{b}}Y^{\underline{a}}+ig_2(\partial_{\underline{a}}W^-{}_{\underline{b}}-\partial_{\underline{b}}W^-{}_{\underline{a}}) \times W^{+,\underline{b}}Y^{\underline{a}}-ig_2(W^-{}_{\underline{a}}W^+{}_{\underline{b}}-W^-{}_{\underline{b}}W^+{}_{\underline{a}})\partial_{\underline{a}}Y_{\underline{b}}. \tag{A4}$$

The quartic couplings are given by

$$\mathcal{L}_q=-g_2^2\,W^+\cdot W^-\,Y\cdot Y+g_2^2\,W^+\cdot Y\,W^-\cdot Y + \frac{g_2^2}{2}(W^+\cdot W^+W^-\cdot W^- - W^+\cdot W^-W^-\cdot W^+). \tag{A5}$$

There are also the contributions containing explicit couplings to the Higgs particle. With the ϕ^4 potential we obtain,

$$\mathcal{L}_\phi=\frac{1}{2}\partial_{\underline{a}}\phi\partial^{\underline{a}}\phi-\frac{1}{2}m_\phi^2\phi^2-\frac{\lambda}{4}(4v\phi^3+\phi^4)+\frac{g^2}{4}(2v\phi+\phi^2) \times W^+\cdot W^-+\frac{1}{8}(g_1^2+g_2^2)(2v\phi+\phi^2)Z\cdot Z \tag{A6}$$

with a mass $m_\phi^2=2\lambda v^2$. The doublet fermions may be added to the bosonic sector described above; we do not list their couplings.

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